Journal Of Investment Management, Vol. 18, No. 4, (2020), pp. 94–129

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MULTI-PERIOD PORTFOLIO SELECTION: A PRACTICAL SIMULATION-BASED FRAMEWORK

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The topic of optimal portfolio selection over time has garnered significant attention from investment researchers since the introduction of portfolio theory in 1952. *While computational, theoretical, and numerical methods have advanced, solutions introduced to date have yet to effectively address many practical aspects of the multi-period portfolio selection problem.*

In this paper, we propose three key requisites for practical multi-period portfolio selection solutions that highlight the central challenges of managing portfolios across a multi-period investment horizon: effective duration management, incorporating real-world asset dynamics, and considering investment frictions and illiquidities. Based on these criteria, we detail an analytical framework for multi-period portfolio selection that provides intuition and yields guiding principles that describe how allocations and duration should evolve across a multi-period investment horizon, given specific investor objectives. We then introduce a practical simulation-based portfolio selection (SBPS) framework and present solutions for common investor objectives that are not only aligned with intuitive principles but also demonstrate the flexibility afforded by SBPS in allowing us to address the three stated requisites for practical multi-period solutions.

1 Introduction

A key distinction between single-period and multi-period portfolio selection is the consideration of intermediate actions within an investment horizon that extends over many periods. In the single-period setting, an investor decides how to invest at the beginning of the period and then waits until the end of the period to assess the outcome. In practice, investing is not so straightforward. Investors with long investment horizons, such as pension fund managers or individuals planning for retirement income, fund their portfolios over

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time and can have multiple objectives across a multi-stage investment horizon that includes a variety of intermediate actions.

There are many reasons why an investor might be required to make intermediate decisions within their investment horizon. These can include cash inflows and/or outflows, changes in expectations, changes in investment opportunities, rebalancing to maintain portfolio risk characteristics, and changes in risk aversion. Investors generally address many of these intermediate decisions by updating their portfolios periodically. In practice, investors commonly determine their optimal portfolio allocations using long-term (10-year) return, risk, and correlation forecasts. They then proceed to update their allocations every 3 to 5 years. This results in investors implementing a sequence of single-period optimal portfolios, where no single period allocation is ever held to the end of the period for which it was intended.

While this type of approach allows for the consideration of important and practical aspects of investing over long horizons, it generally results in less than optimal allocations, unnecessary costs, and a failure to consider key aspects of how to most efficiently evolve portfolio allocations through time. This is so because the approach does not explicitly account for intermediate actions or intertemporal allocation decisions that have implications that extend across multiple periods. Multi-period portfolio selection seeks to address this by determining efficient time-varying portfolio allocations across the investment horizon in the context of specific investor objectives while considering pre-determined intermediate decisions.

Initial research on multi-period portfolio selection by Mossin (1968), Merton (1969), and Samuelson (1969) focused on dynamic programming solutions as originally suggested by Markowitz (1959). Much of the subsequent research has followed in the same vein and has served to advance theory but has been significantly limited in its practical application by what Richard Bellman, who introduced dynamic programming in 1953, described as "the curse of dimensionality." This explains that computational requirements increase exponentially as the number of state variables considered in solving a dynamic programming problem increase. While computational, theoretical, and numerical methods have since evolved to a point where dynamic programming approaches can now be used to provide more practical solutions, many important aspects of the multi-period problem have yet to be addressed.

In this paper, we introduce a simulation-based portfolio selection (SBPS) framework that addresses central challenges of implementing and managing portfolios across a multi-period investment horizon: effective duration management, incorporating real-world asset dynamics, and considering investment frictions and illiquidities. The framework produces multi-period solutions that align portfolio durations with expected cash flows and allows for the inclusion of individual hold-to-maturity and defined-maturity investments alongside traditional assets, such as duration-targeted fixed-income investments, regularly used for strategic asset allocation.[1]

We have organized the paper as follows. Section 1 provides a brief literature review of multi-period and related research. Section 2 proposes three key requisites for practical multi-period portfolio selection solutions and provides supporting discussion for each. Section 3 sets up the multi-period problems on which we will be focusing by presenting the objective functions for two common investor types: Growth investors, who seek to maximize terminal wealth, and Income investors, who seek to maximize a series of future cash flows. We also distinguish between growth and duration assets. In Section 4 we detail an analytical framework that considers the distinction between growth and duration assets and produce analytical solutions for Growth and Income investor objectives. We consider scenarios when no cash flows are expected and when both cash inflows and outflows are expected for cases when only growth assets are considered as well as when both growth and duration assets are considered. We present the analytical solutions and assess the sensitivities of allocations and durations to correlations between assets and to different yield curve slopes. We then share key observations from our assessment of the analytical solutions that will then serve as guiding principles for how we should expect allocations to evolve over a multi-period investment horizon, given specific investor objectives. In Section 5, we introduce a practical simulation-based portfolio selection framework and present multi-period solutions that are not only aligned with the principles developed as part of our analytical exercise but also demonstrate the flexibility afforded by SBPS in allowing us to address the three stated requisites for practical multi-period solutions. We conclude in Section 6.

2 Multi-period Portfolio Selection theory and related literature

Markowitz's (1952) *Portfolio Selection* paper forever changed the practice of investment management by introducing a framework for single-period portfolio selection. In 1959, Markowitz presented a much broader exposition of the framework and also provided guidance on the problem of portfolio selection through time. Here he explained how, under certain conditions, a dynamic programming approach could be used to provide an exact solution.[2] He conceded, however, that dynamic programming techniques were probably infeasible due to the computational requirements of even the simplest of utility functions.[3]

Over the following decade, multi-period portfolio selection theory advanced notably with Mossin's (1968) and Samuelson's (1969) discrete-time dynamic programming models and Merton's (1969) continuous-time counterpart. An interesting and counterintuitive result of this initial work was the concept of myopic portfolio choice. This was the special case where a multi-period investor would hold the same portfolio as that of a single period investor. This result indicated that a multi-period investor would not become more cautious as they approach retirement. This special case, however, is a function of a particular set of simplifying assumptions.[4] Research from this point forward generally explored the conditions under which myopic portfolio choice held. Of particular interest was the work by Hakansson (1971) that discussed the implications of serial correlations in yields on multi-period portfolio choice.

Following this initial work, many of the subsequent advances in multi-period theory were again predominantly based on dynamic programming. Campbell and Viceira (2002) provide an excellent overview of advances in multi-period portfolio selection as well as discussion regarding the many challenges faced by long-horizon investors. This work also includes an insightful discussion and guidance regarding the incorporation of specific types of bonds for long-horizon investors. Along these lines, Fisher and Weil (1971) expand on Macauley's (1938) work on understanding the implications of changes in the yield curve on bond portfolio values and demonstrate that fixed-income instruments have important intertemporal relationships. Langetieg *et al*. (1990) discuss "partial immunization" of duration risk using duration-targeted fixed-income strategies across multi-period horizons. Liebowitz *et al*. (2014) expand on the distinctive nature of duration-targeted investments and implications for those types of investment across multi-year investment horizons.

As multi-period research continued, thinking on other important aspects of long-horizon investing was advancing. Markowitz (1991) proposed a "Game of Life" simulation for modeling the complex investment planning problems faced by individual investors. Markowitz and van Dijk (2003) presented a dynamic programming approximation that could allow for the efficient reallocation of portfolios over time and changing market conditions. Wilcox *et al*. (2008) present a framework for considering taxes as part of the portfolio management process. Hoevenaars *et al.* (2009) explore the implications of incorporating alternative asset classes into long-horizon portfolios. Finally, Blay and Markowitz (2016) and Markowitz (2016) present a simulation-based Net Present Value portfolio analysis approach that explicitly accounts for the impact of taxation on assets invested across a multi-period investment horizon.

While thinking was advanced on many aspects of long-horizon investing, the practical implementation of dynamic programming approaches originally proposed would remain well beyond computational capabilities until recently when a combination of advances in theory and computational power has allowed for the computation of solutions of greater relevance to investors. Dempster and Medova (2011) presents an individual asset liability management model for life cycle planning using dynamic stochastic optimization that also considered the implications of taxation on assets. Das *et al*. (2018, 2019) present dynamic programming approaches (without tax considerations) to solving the multi-period portfolio selection problem with both single and multiple objectives.

The simulation-based multi-period portfolio selection framework proposed in this paper builds on much of the thinking to date on long-horizon multi-period investing but diverges in that it does not employ dynamic programming. Instead, we propose a much more flexible framework that decomposes the multi-period problem into three distinct parts: the objective function, simulation, and optimization. This not only provides substantial flexibility in addressing the central challenges of implementing and managing portfolios across a multi-period investment horizon, it also facilitates the advancement of multi-period portfolio selection research as innovations in any of the three areas can easily be incorporated into the proposed framework. This stands in contrast to dynamic programming approaches where consideration of additional real-world aspects of multi-period portfolio management likely requires a non-trivial reformulation of the solution to be used.

3 Requisites for practical multi-period Portfolio Selection

As computational power, theory, and numerical methods have evolved, so too must multi-period investing solutions evolve to account for the practical realities involved in implementing and managing portfolios across long investment horizons. In this section, we propose three requisites for practical multi-period portfolio selection solutions.

3.1 Solutions must evolve allocations and duration over time to align with expected cash flows

The multi-period portfolio selection problem differs most notably from the single period problem in that it typically considers long investment horizons and generally includes expected cash inflows and outflows. The implications of this must be reflected in both the evolution of allocations and in the duration profiles of proposed solutions across the investment horizon.

To demonstrate the importance of the above, we examine what constitutes a risk-free asset. In the single period context, cash or US Treasury bills are often considered risk-free assets due to their low volatility. However, over long investment horizons, these assets expose investors to meaningful uncertainty. Longer duration bonds held in the context of a duration-targeted index fund also experience a similar challenge as they remain exposed to a relatively constant amount of interest rate risk through time. Allowing for a time-varying duration profile can address the limitations of both these alternatives. Consider an investor needing to fund a liability in 10 years. Figure 1 presents 20 simulated paths for three alternative approaches: investing in a money market fund, investing in a duration-targeted index fund of Treasury bonds with a duration of 10 years and investing in a 10-year zero coupon Treasury bond. As can be seen, despite exhibiting meaningful volatility early on, the zero-coupon Treasury bond, with its accompanying declining duration through time resulted in the lowest risk solution.

This leads to the important conclusion that effective duration management is an important aspect of multi-period solutions. In cases like the one above, hold to maturity investments can be the most efficient assets to employ. Alternatively, or in more complex cases, time-varying combinations of short- and long-duration bond funds can also be used to achieve the desired behavior.

3.2 *Solutions must consider real-world asset dynamics*

Similar to the portfolio construction process, asset pricing dynamics need to also reflect some of the unique characteristics of long-horizon investing. These asset price characteristics can materially impact how theoretical solutions actually perform when implemented in the real world. There is ample evidence for this view in the literature. Looking at equities, Poterba and Summers (1988) consider whether prices are mean-reverting, using data from the United States and 17 other countries. Their estimates imply positive autocorrelation in returns over short horizons and negative autocorrelation over longer horizons. Spierdijk *et al.* (2012) present evidence of mean reversion for 17 developed economies. Spierdijk and Bikker (2012) conclude that if stock prices are mean-reverting, stocks are relatively less risky for longer investment horizons, so that a larger share of wealth may be allocated to stocks. The same is true if stock returns show negative autocorrelation, which is often referred to in the literature as mean reversion in stock returns. Kinlaw *et al.* (2014) discuss the misestimation of both risks

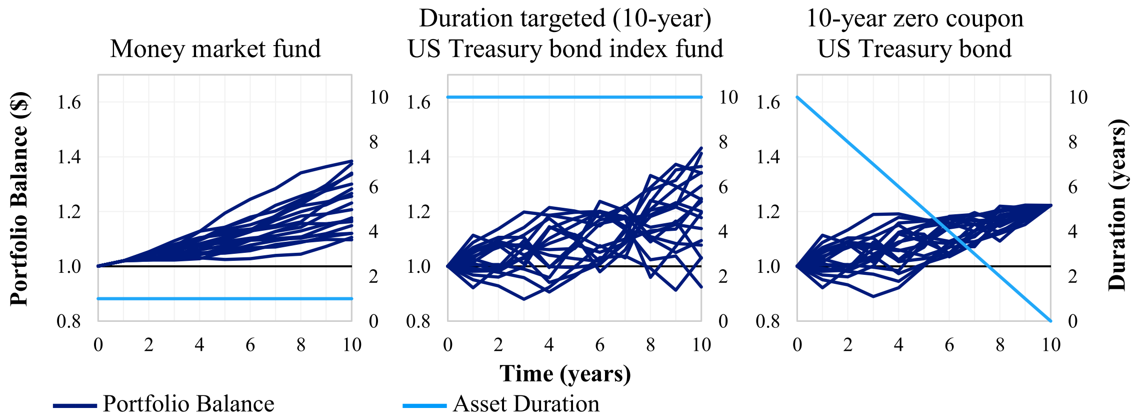

Figure 1 Twenty simulated paths representing growth of $1 invested in a money market fund, a rebalanced 10-year constant duration treasury bond index and a 10-year zero-coupon bond held to maturity. The duration through time of these investments is plotted against the right axis.

and correlations that result from the practitioner assumption that asset prices are independently and identically distributed (i.i.d.). They explain that these misestimations are a direct result of the fact that financial time series often exhibit serial dependence, mean reversion, trending, and/or risk clustering. Davis (2014) argues that investment consulting firms that provide stochastic modeling seem to believe in mean reversion, since almost all of them regularly adjust their assumption for future stock returns based on some measure of stock valuation levels (e.g., dividend yield, or P/E ratios).

Effective duration management and the incorporation of individual fixed-income instruments also require that the modeling of fixed-income securities aligns with prevailing interest rate dynamics. For example, assuming normal or lognormal distributions for fixed-income security returns in a low rate environment may imply some unrealistic scenarios. For such instruments, it is important to first model yield curve movements and then price securities off the resulting curves. The modeling of the yield curve itself needs to also align with various foundational assumptions. For example, some models do not allow for negative interest rates, while others allow for rates to go only slightly negative in line with what has been recently observed in various economies.

3.3 Solutions must consider investment frictions and illiquidities

In practice, investing is rarely as simple as making a single initial investment in a portfolio that is held until the end of an investment period. Real-world portfolios implemented across multiple periods are regularly updated for a variety of reasons. This includes updates resulting from cash inflows, cash outflows, the purchase, sale or replacement of funds or securities, portfolio rebalancing, and changes in preferences. All these intermediate portfolio management decisions can have significant implications for investing outcomes. For example, transaction fees can erode portfolio balances over time especially if they are high and/or frequent. Trading can also result in tax consequences. Taxation significantly increases the complexity for multi-period solutions due to the path- and time-dependent nature of taxation.[5] Over the short term, these real-world elements of investing may only result in negligible differences in expected outcomes. Over the long term, however, significant differences between actual and expected outcomes can arise from not adequately accounting for the impact of these elements. Consequently, practical solutions that must ultimately distinguish between successful and unsuccessful outcomes must consider the impact of these real-world aspects of investing if they are to provide investors with reliable information that can be used to support investment decision-making for long-horizon investing.

4 Setup

The general multi-period portfolio selection problem can be summarized as consisting of an initial investment followed by a series of funding cash inflows made during an accumulation period that are then followed by a series of consumption cash outflows during a consumption period. The size and timing of cash flows can vary. Funding a bequest at the end of the investment horizon might also be a consideration. In short, investors may be concerned with funding consumption cash flows, the amount of terminal wealth or both. We can describe the problem more precisely using the following notations:

- The investor has an investment horizon of T periods indexed by $t = 0, 1, \ldots, T$
- The investor has access to N assets
- The $N \times 1$ random vectors $\boldsymbol{r}_t$ for $t = 0, 1, \ldots, T-1$ represents the asset returns observed between time t and $t+1$

- The investor has an initial capital C_0 at time $t = 0$. Additionally, the investor will realize a series of net cash flows $C_1, C_2, \ldots, C_T$ in future at time $t = 1, 2, \ldots, T$. Cash inflows are positive whereas outflows are negative
- The investor follows an investment strategy through time given by the sequence of $N \times 1$ portfolio weight vectors $\boldsymbol{w}_t$ for $t = 0, 1, \ldots, T-1$. We represent the investment strategy by $\boldsymbol{W} = (\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{T-1})$
- The portfolio wealth through time is P_t for $t = 0, 1, \ldots, T$; the starting portfolio wealth P_0 is the same as the initial capital C_0

Portfolio wealth evolves through time recursively as $P_{t+1} = P_t(1 + w_t^T r_t) + C_{t+1}$. Notice that P_{t+1} depends on the weight vectors or investment decisions up to time t. Thus, P_{t+1} is a function of $(\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_t)$. In particular, the terminal portfolio wealth P_T is a function of all the weight vectors up to time $T - 1$ (i.e., a function of $\boldsymbol{W} = (\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{T-1})$). We can expand the recursive nature of the portfolio wealth to write the terminal portfolio wealth as:

$$P_T(W) = P_T = \sum_{t=0}^{T-1} C_t(1 + w_t^T r_t) \ldots (1 + w_{T-1}^T r_{T-1}) + C_T. \tag{1}$$

We will next consider the objectives of two types of investors: Growth investors and Income investors. Below we detail the objective functions we use for each of these investor types.

4.1 The growth investor's objective

The growth investor seeks to maximize terminal wealth while reducing the uncertainty around terminal wealth outcomes. The growth objective seeks to maximize the objective function:

$$J(\boldsymbol{W}) = Q_\alpha(\boldsymbol{W}) - \lambda(Q_\alpha(\boldsymbol{W}) - Q_\beta(\boldsymbol{W})). \tag{2}$$

where $Q_\alpha(\boldsymbol{W})$ and $Q_\beta(\boldsymbol{W})$ are the α-th and β-th quantiles of the probability distribution of the terminal wealth $P_T(\boldsymbol{W})$. In particular, we view $Q_\alpha(\boldsymbol{W})$ as a percentile-based measure of central tendency, i.e., the median of $P_T(\boldsymbol{W})$. This is an important distinction for the distribution of long-horizon investing outcomes which tend to be skewed with fat right tails that can bias the mean value of terminal wealth. We view $Q_\beta(\boldsymbol{W})$ as a percentile-based risk measure such as the 5th percentile of $P_T(\boldsymbol{W})$. The parameter $\lambda > 0$ is a control parameter that balances the reward (median) against the risk (median minus the 5th percentile). This describes an investor seeking to maximize median wealth with an aversion to unfavorable terminal wealth outcomes.

4.2 The Income investor's objective

The Income investor seeks to maximize a series of consumption cash flows for a given probability of remaining solvent. This describes a multi-period income maximization problem. Given a pre-defined cash flow schedule $C_0, C_1, \ldots, C_T$, which can include both inflows and outflows, we consider an investor who seeks to make a series of additional withdrawals A, over and above the pre-defined cash flows. The income investor seeks to consume the additional amount A starting at $t = T_A$ until $t = T$, where $t = T_A$ is the time at which the additional consumption begins. The set of cash flows for the investor are now $C_0, C_1, \ldots, C_{T_A} - A, \ldots, C_T - A$. Instead of maximizing the income withdrawal directly, we maximize the probability of staying solvent up to time $t = T$ subject to the additional withdrawals. The income investor seeks to maximize the objective:

$$J(\boldsymbol{W}) = Prob(\tau_A(\boldsymbol{W}) \geq T + 1), \tag{3}$$

where $\tau_A(\boldsymbol{W})$ is the timing of the insolvency event otherwise described as the discrete point in time when wealth as a function of $\boldsymbol{W}$ and A falls below zero for the first time. The formulation of this objective function describes an investor seeking to maximize the probability of remaining solvent up to time $t = T$ subject to a certain amount of additional periodic withdrawals.

In Section 6, we present a simulation-based portfolio selection (SBPS) approach for the above two variations in the multi-period problem. However, we first provide approximate analytical solutions to these multi-period problems in the next section. This will allow us to develop intuition behind the dynamics of multi-period solutions as well as help us explain the results from our simulation-based portfolio selection approach.

5 An analytical multi-period Portfolio Selection framework

To develop intuition as to how portfolio allocations might be expected to evolve across an investor's investment horizon, we begin by presenting an analytical solution to the *Growth investor's* problem. We then proceed to show that the *Income investor's* problem can be solved by solving a sequence of growth investor problems — a key duality principle.

Furthermore, to address the problems analytically, we will explore the first order drivers of portfolio wealth which drive optimal solutions. Observe that for small portfolio returns $r_1, r_2, \ldots, r_n$ we can approximate the compounded return $(1 + r_1)(1 + r_2)\ldots(1 + r_n)$ as $1 + r_1 + r_2 + \cdots + r_n$. Using this approximation, we can write the first order approximation $P_T^{(1)}(\boldsymbol{W})$ of the terminal wealth $P_T(\boldsymbol{W})$ as:

$$
\begin{aligned}
P_T^{(1)}(\boldsymbol{W}) &= \sum_{t=0}^{T-1} C_t(1 + \boldsymbol{w}_t^T \boldsymbol{r}_t \\
&\quad + \cdots + \boldsymbol{w}_{T-1}^T \boldsymbol{r}_{T-1}) + C_T \\
&= \sum_{t=0}^{T-1} X_t \boldsymbol{w}_t^T \boldsymbol{r}_t + X_T, \qquad (4)
\end{aligned}
$$

where $X_t = C_0 + \cdots + C_t$ are the partial sums of the cash flows for $t = 0, 1, \ldots, T$.

With this first-order approximation of terminal wealth, we are ready to explore the analytical solutions to the two types of investor problems described in the previous section. In Section 5.1, we focus on solving for the Growth investor's objective and distinguish between growth and duration assets and then derive solutions for **Growth-Only** (Section 5.1.1) and **Growth + Duration** (Section 5.1.2) asset portfolio cases. In Section 5.1.3, we explore the behavior of the solutions for these two cases and assess the impact of cash flows, asset correlations, and interest rate characteristics on the solutions which allows us to intuit key aspects of multi-period portfolio selection. Finally, in Section 5.2, we discuss a duality principle that allows us to view the solution to the income investor's objective as a solution to the growth investor's objective.

5.1 Analytical formulation of the growth investor objective

A key limitation of a median-percentile objective is the lack of analytical tractability of the derivatives. The mean–variance objective overcomes this limitation by allowing us to write down derivatives analytically and thus, serves as a natural proxy of the median-percentile objective in our analytical framework. We seek to maximize the following mean–variance objective function for the first-order terminal wealth:

$$J(\boldsymbol{W}) = E(P_T^{(1)}(\boldsymbol{W})) - \lambda \text{ Variance}(P_T^{(1)}(\boldsymbol{W})).$$

In our analytical framework, we consider two distinct types of assets that we define as growth and duration assets:

Growth Assets are assets that produce returns primarily through capital appreciation. These types of assets often have the potential for higher investment returns over the longer term, but they also tend to exhibit higher investment risk. Examples of growth assets include individual stocks, equity funds, commodities, multi-asset funds, private real estate, and private equity investments.

Duration Assets are assets that produce returns primarily through income payments rather than capital appreciation. For the purposes of this paper, we will limit such assets to government issued securities with no embedded default risk. Examples include Treasury bills, Treasury bonds, zero-coupon bonds, and TIPS. If held to maturity, the return of these instruments is known with virtual certainty. Prior to maturity, these instruments also exhibit duration risk, meaning that their price and corresponding periodic returns are directly linked to changes in interest rates.

To provide insight into the dynamics of the multi-period mean–variance problem we will first consider solutions for portfolios composed exclusively of growth assets and then extend our results to solutions for portfolios composed of both growth and duration assets.

5.1.1 *Growth asset portfolios (Growth-Only)*

We first consider the case when the investment opportunity set consists exclusively of growth assets. Let the return vectors of the growth assets be $\boldsymbol{r_t}$ for $t = 0, 1, \ldots, T-1$ which are assumed to be independently and identically distributed (i.i.d.) normal random variables with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The optimal weight vector $\boldsymbol{w_t}$ is:

$$\boldsymbol{w_t} = \begin{pmatrix} \boldsymbol{I} \\ -\boldsymbol{1}^T \end{pmatrix} \boldsymbol{A}^{-1} \left(\frac{1}{2\lambda X_t} \boldsymbol{m} + \boldsymbol{b} \right) + \boldsymbol{e}_N, \tag{5}$$

where $\boldsymbol{I}$ is the $N-1 \times N-1$ identity matrix, $\boldsymbol{1}$ is the $N-1 \times 1$ vector of ones, and $\boldsymbol{e}_N$ is the $N \times 1$ index vector with one at the N-th position and zero everywhere else. The $N-1 \times 1$ vector $\boldsymbol{m}$ is the vector of excess returns of the first $N-1$ assets over the N-th asset, and the $N-1 \times N-1$ matrix $\boldsymbol{A} = (\boldsymbol{I}\ \ -\boldsymbol{1})\boldsymbol{\Sigma}(\boldsymbol{I}\ \ -\boldsymbol{1})^T$ is the relative covariance matrix of the first $N-1$ assets with respect to the N-th asset, and finally the $N-1 \times 1$ vector $\boldsymbol{b} = (-\boldsymbol{I}\ \boldsymbol{1})\boldsymbol{\Sigma}\boldsymbol{e_N}$ captures the diversification benefit of the first $N-1$ assets with the N-th asset.

See Appendix A for proof of the above formula.

5.1.2 *Growth and duration asset portfolios (Growth + Duration)*

We can now move to consider the case where the investor has access to one growth asset and one duration asset. The investor can choose a duration asset of any duration at each point in time. We seek to determine:

- the optimal allocation between the growth and duration assets
- the optimal duration of the duration asset chosen in each period

We assume that a simplistic yield process drives the price of the duration assets. The discrete time instantaneous yield process is given by:

$$y_t = y_0 + \Delta y_0 + \Delta y_1 + \cdots + \Delta y_{t-1},$$

where Δy_t are independently and normally distributed as $N(\mu_{y,t}, \sigma_y^2)$ for $t = 0, 1, \ldots,$ and the time dependent parameter $\mu_{y,t}$ represents the drift of the instantaneous yield process and can be used to describe a changing rate environment such as increasing or decreasing or flat. The parameter σ_y captures the volatility of the instantaneous yield process. Additionally, at any time t, the yield curve is linear with a total slope s up to the maximum maturity $T_{\max}$. Schematically the yield curve at time t is represented below.

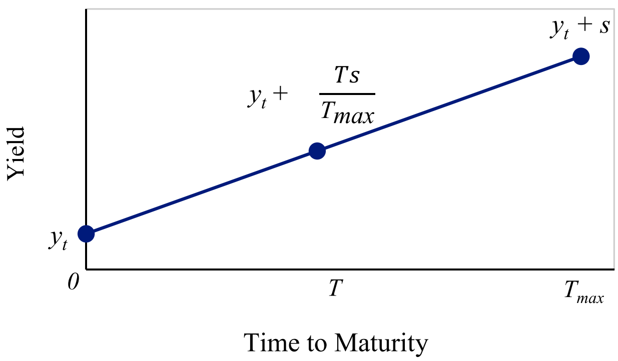

Figure 2 The yield curve at time t. Here, the instantaneous yield is y_t and the $T_{\max}$ maturity bond has a yield of $y_t + s$. In particular, a $T < T_{\max}$ maturity bond has a yield of $y_t + Ts/T_{\max}$.

Under the above process, the yield of the T-maturity bond at time t is given by:

$$y_t^{(T)} = y_t + \frac{Ts}{T_{\max}}.$$

In addition to the yield process, we describe the growth asset return through time r_t for $t = 0, 1, \ldots$ as i.i.d. $N(\mu_g, \sigma_g^2)$, and at any time t the growth asset return r_t and the change in instantaneous yield Δy_t has a correlation $\rho_{g,y}$, i.e., $\mathrm{cov}(r_t, \Delta y_t) = \rho_{g,y}\sigma_g\sigma_y$. The correlation between the duration asset and the growth asset is approximately $\rho_{d,y} = -\rho_{g,y}$.

Using this setup, we provide the analytical solution for the optimal growth asset allocation and the optimal duration through time as the solution to the mean–variance problem. Let $\boldsymbol{w} = (w_0, w_1, \ldots, w_{T-1})$ represent the growth asset weights and $\boldsymbol{D} = (D_0, D_1, \ldots, D_{T-1})$ the duration of the duration asset through time. The mean–variance objective function as shown in Appendix B is not a quadratic function in the unknowns $\boldsymbol{w}$ and $\boldsymbol{D}$. However, if the durations $D_0, D_1, \ldots, D_{T-1}$ are known, then the objective function is quadratic in $w_0, w_1, \ldots, w_{T-1}$ and vice versa. This allows us to provide an explicit formula for the optimal growth asset allocation when the durations are known and the optimal durations when the growth asset allocations are known.

When the optimal durations $D_0^{(\lambda)}, D_1^{(\lambda)}, \ldots, D_{T-1}^{(\lambda)}$ are known, then the optimal growth asset weights through time $\boldsymbol{w} = (w_0, w_1, \ldots, w_{T-1})$ are:

$$\boldsymbol{w} = \boldsymbol{U}^{-1}\left(\frac{1}{2\lambda}\boldsymbol{m} - \boldsymbol{v}\right) \tag{6}$$

and alternatively, when the optimal growth asset weights $w_0^{(\lambda)}, w_1^{(\lambda)}, \ldots, w_{T-1}^{(\lambda)}$ are known, the optimal duration is:

$$\begin{aligned} D_t = & \underbrace{\sum_{k=0}^{T-1-t} \frac{X_{t+k}(1 - w_{t+k}^{(\lambda)})}{X_t(1 - w_t^{\lambda})}}_{\substack{\textit{cash flow}\\ \textit{contribution}}} \\ & + \underbrace{\frac{\rho_{g,y}\sigma_e w_t^{(\lambda)}}{\sigma_y(1 - w^{(\lambda)})}}_{\substack{\textit{correlation}\\ \textit{contribution}}} \\ & - \underbrace{\frac{\mu_{y,t}}{2\lambda X_t(1 - w_t^{(\lambda)})\sigma_y^2}}_{\textit{yield drift contribution}} \\ & + \underbrace{\frac{s}{\lambda X_t T_{\max}(1 - w_t^{(\lambda)})\sigma_y^2}}_{\substack{\textit{slope}\\ \textit{contribution}}}, \end{aligned} \tag{7}$$

where $\boldsymbol{\mu_D}$ is the mean return vector of the duration asset through time (detailed in Appendix B) and the vector $\boldsymbol{m} = \mu_g \boldsymbol{X_d} \boldsymbol{1} - \boldsymbol{X_d}\boldsymbol{\mu_D}$ is the vector of dollar-weighted excess returns of the growth asset relative to the duration asset. The vector $\boldsymbol{v} = (\rho_{g,y}\sigma_g\sigma_y \boldsymbol{X_d}\boldsymbol{X_u} - \sigma_y^2 \boldsymbol{X_u^T}\boldsymbol{X_u})\boldsymbol{1}$ captures the dollar-weighted diversification benefit of the duration asset with the growth asset. Finally, the matrix $\boldsymbol{U} = \sigma_g^2 \boldsymbol{X_d^2} + \sigma_y^2 \boldsymbol{X_u^T}\boldsymbol{X_u} - 2\rho_{g,y}\sigma_g\sigma_y\boldsymbol{X_d}\boldsymbol{X_u}$ is defined in terms of the diagonal matrix $\boldsymbol{X_d}$ and the upper triangular matrix $\boldsymbol{X_u}$.

$$\boldsymbol{X_d} = \begin{pmatrix} X_0 & 0 & \cdots & 0 \\ 0 & X_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{T-1} \end{pmatrix} \quad \text{and}$$

$$\boldsymbol{X_u} = \begin{pmatrix} X_0 & X_1 & \cdots & X_{T-1} \\ 0 & X_1 & \cdots & X_{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{T-1} \end{pmatrix} - \operatorname{diag}(\boldsymbol{D})\boldsymbol{X_d}.$$

See Appendix B for proofs of the above formulas.

5.1.3 Analytical examples and key insights

We are now ready to explore analytical solutions to the Growth investor problem with two multi-period scenarios, one with no cash flows and one with cash inflows and outflows:

Scenario 1: Maximize terminal wealth (no cash flows)

- A 10-year investment horizon
- An initial investment of $1,000
- No cash flows

Scenario 2: Maximize terminal wealth (with cash inflows and outflows)

- A 10-year investment horizon
- An initial investment of $1,000
- Annual inflows of $500 over the first 5 years and annual outflows of $500 over the last 5 years

For each scenario we consider the following two cases distinguished by the assets made available for portfolio construction:

- **Growth-Only:** we consider two growth assets; a high growth asset with a mean return $\mu_1 = 6\%$ and volatility $\sigma_1 = 15\%$ and a low growth asset with a mean return $\mu_2 = 2\%$ and volatility $\sigma_2 = 4\%$
- **Growth+Duration:** we consider the high growth asset and a duration asset priced off of a yield process starting at initial yield $y_0 = 2\%$ with no drift and volatility $\sigma_y = 1\%$.

Growth-Only and Growth+Duration efficient frontiers are produced for each of these scenarios by maximizing the objective:

$$E(P_T^{(1)}(\boldsymbol{W})) - \lambda \text{ Variance}(P_T^{(1)}(\boldsymbol{W})),$$

for a range of λ values with $\lambda \geq 0$. To facilitate our understanding of the evolution of portfolio allocations and portfolio durations across multi-period investment horizons, we consider a single solution from each of the frontiers. To best demonstrate the dynamics of multi-period portfolio selection we focus on mid-frontier solutions. For Growth-Only frontiers we select solutions with a 50%/50% average allocation to low and high growth assets. For Growth+Duration frontiers we select solutions with a 50%/50% average allocation to growth and duration assets.

Furthermore, we analyze the sensitivity of the Growth-Only allocations to the growth asset correlations ($\rho = -1$, 0 and 1) and the sensitivity of Growth+Duration allocations to the growth-duration asset correlations ($\rho_{g,d} = -\rho_{g,y} = -0.25$, 0 and 0.25) and yield curve slope ($s = 2\%, 0\%$ and -2%). Figures 3 and 4 present cash flows, Growth-Only allocations through time, and Growth+Duration allocations and durations through time for the two growth investor scenarios.

Based on the analysis and the examples presented, we make the following key observations in the context of a growth investor's terminal wealth maximization objective.

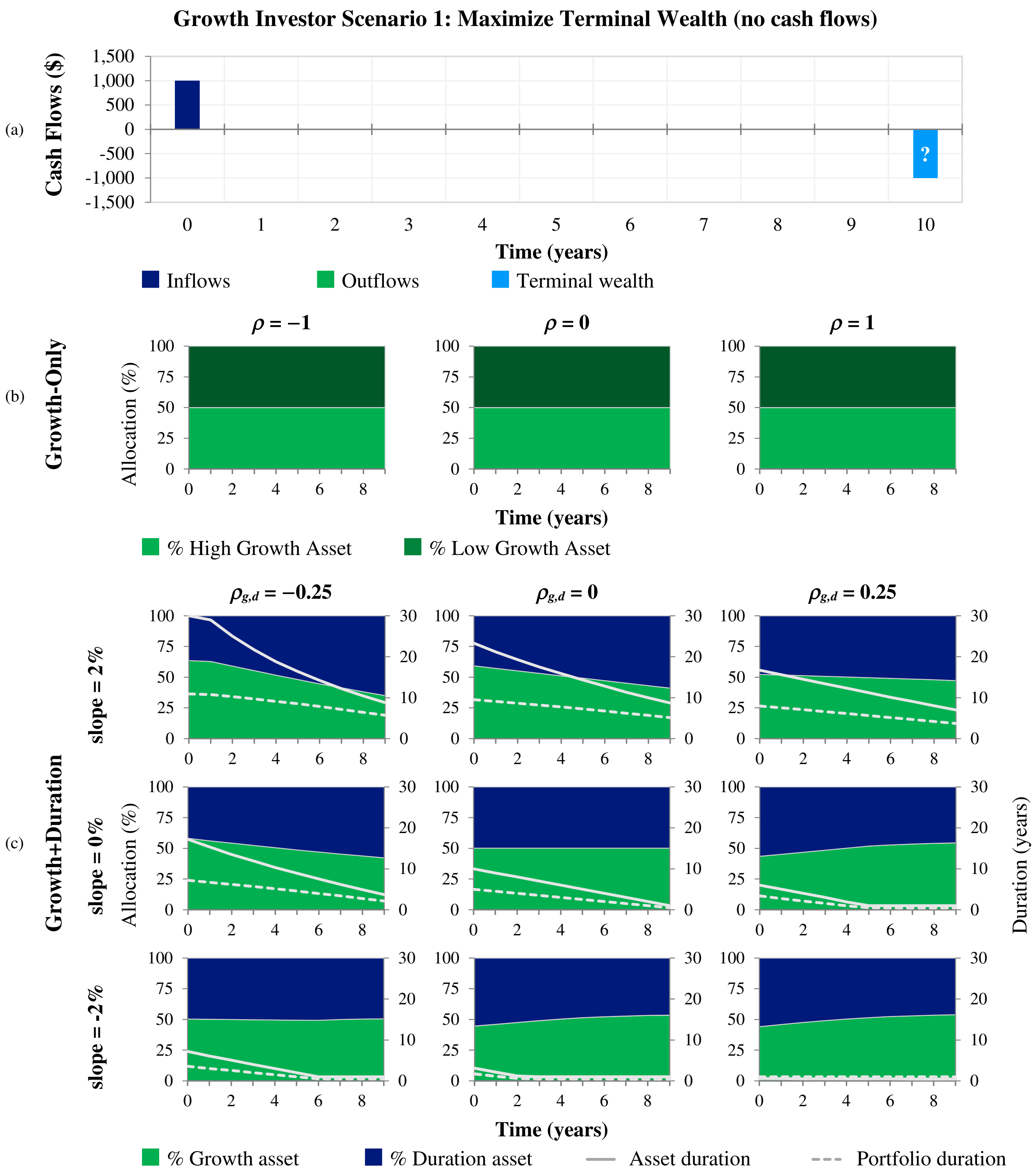

Figure 3 **(a)** Scenario cash flows. Terminal wealth at $t = 10$ to be maximized **(b)** Growth-Only allocations for three asset correlation scenarios. **(c)** Growth + Duration allocations and durations for nine (3 × 3) asset correlation and yield curve slope combinations.

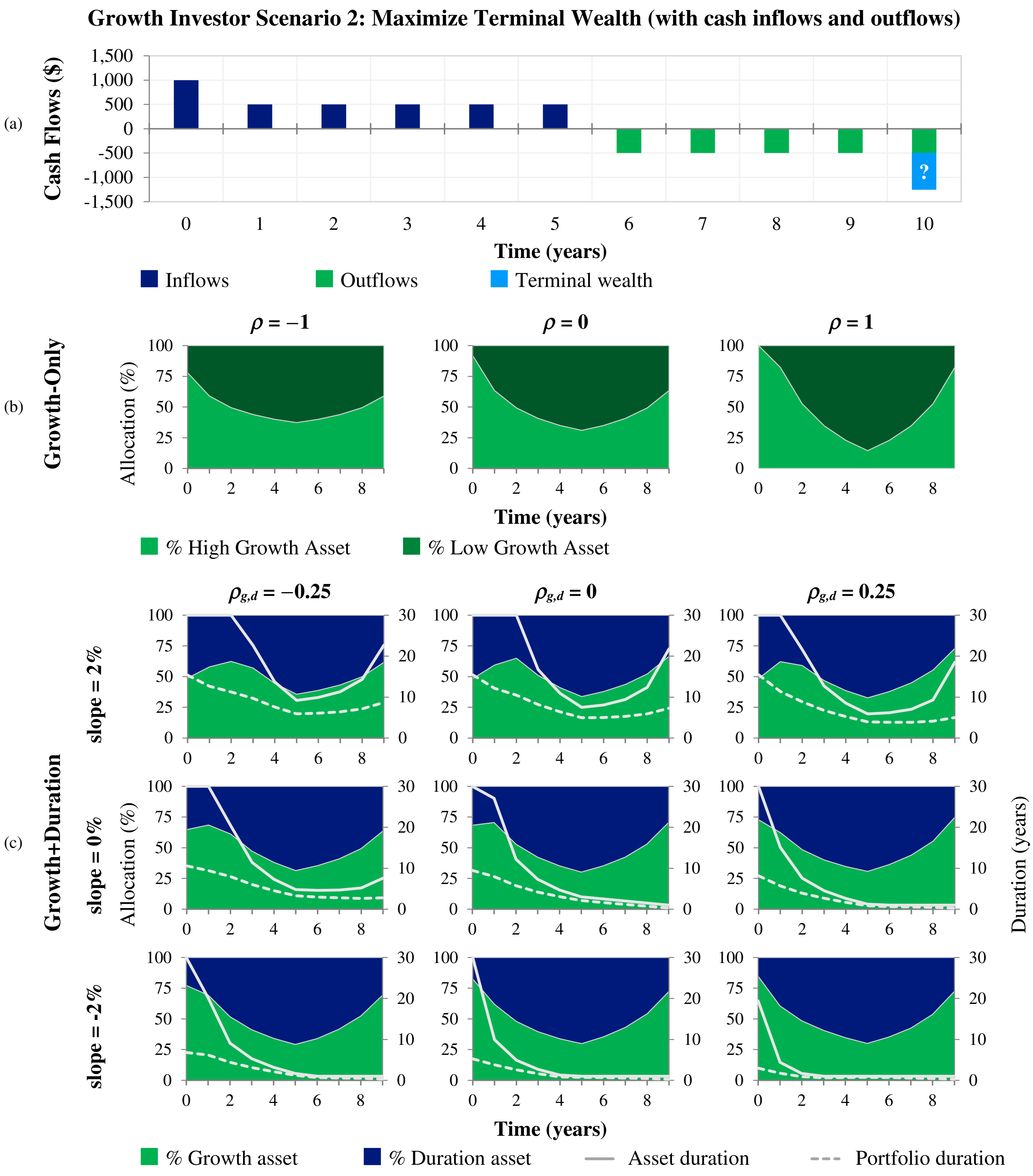

Figure 4 **(a)** Scenario cash inflows/outflows. Terminal wealth at $t = 10$ to be maximized **(b)** Growth-Only allocations for three asset correlation scenarios. **(c)** Growth + Duration allocations and durations for nine (3×3) asset correlation and yield curve slope combinations.

Solutions should shift to safer (riskier) allocations as inflows (outflows) occur

Examining the Growth-Only portfolios in Figure 3, we observe that in the absence of flows, allocations between low-risk and high-risk assets remain unchanged through time. This is a direct result of the commutative nature of how portfolios deliver performance. Good or bad, performance will have the same impact on the terminal wealth, irrespective of whether it occurs earlier or later. However, when we introduce inflows/outflows in Figure 4, things change. Here the optimal solution moves from high-risk to lower-risk during the inflow period and back to higher risk during the outflow period. To better understand why this is so, we first re-write the asset weights through time in Equation (5) for the two-asset case. The optimal weights of assets 1 and 2 are:

$$w_{1,t} = \frac{1}{2\lambda X_t} \frac{\mu_1 - \mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} + \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

and

$$w_{2,t} = -\frac{1}{2\lambda X_t} \frac{\mu_1 - \mu_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2} + \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.$$

The lowest risk allocations through time namely, $w_{1,t}^{(LR)}$ and $w_{2,t}^{(LR)}$ are obtained from the above formulas by substituting $\lambda = \infty$. Observing that $w_{1,t}^{(LR)}$ and $w_{2,t}^{(LR)}$ are constant through time, we will drop the time component from the subscript. We notice that in the above equations, as X_t increases with inflows, the allocation to the high-risk asset decreases; an opposite behavior is observed as X_t decreases with outflows. We can formalize this behavior by saying that for each asset, the dollar allocation over the lowest risk solution is constant over time as shown by:

$$X_t w_{1,t} - X_t w_1^{(LR)} = c(\lambda) \quad \text{and}$$
$$X_t w_{2,t} - X_t w_2^{(LR)} = -c(\lambda),$$

where $c(\lambda)$ is a constant inversely proportional to λ.

This is an important result that indicates that for a fixed horizon mean–variance objective function (and by extension the median–percentile objective function) the timing and magnitude of expected inflows and outflows, to first order, have a significant influence on allocation dynamics. In order to maintain a constant dollar allocation to risk through time, the investor will allocate conservatively when the portfolio balance is high and aggressively when the balance is low, resulting in a U-shaped allocation to risk assets across the investment horizon. It is worth noting that the U-shaped allocation will result as long as the portfolio balance is expected to stay positive (since the above analytical derivation requires X_t, the first-order approximation of portfolio balance, to be positive). However, as the probability of running out of money materializes due to extended or higher withdrawals, allocations shift back to safer assets. The SBPS Income Investor example confirms this behavior as we will see later in Figure 7 (Section 6.4.2). As indicated previously, the SBPS framework is intended to allow for different objective functions. Different objective functions will likely result in different allocation dynamics.

Asset duration should match the investment horizon when there are no cash flows

In Figure 3, focusing on the case with no yield curve slope and no correlation between yield curve movements and growth assets, we observe that the duration of the duration assets mirrors the remaining investment horizon. This behavior can also be seen by further examining Equation (7). More specifically, if we drop the last three terms, in the absence of future cash flows, the duration reduces to:

$$D_t = \sum_{k=0}^{T-1-t} 1 = T - t.$$

In this simple example, the desired duration profile could be implemented via a buy and hold strategy using a duration-matched zero-coupon Treasury bond. It is interesting to note that this result comes in contrast to typical hedging schemes employed by corporate defined benefit plans. Such plans typically employ an LDI framework focused on managing the funding ratio volatility. In doing so, these plans may seek to match the portfolio's *overall* duration to the investment horizon, leading to longer duration targets than are suggested here. While this makes sense in the context of a plan fiduciary, it may be overly restrictive and suboptimal in the context of maximizing an investor's terminal wealth as laid out in this problem.

Duration should be extended (shortened) with expected inflows (outflows)

In Figure 4, where we introduce inflows and outflows, we observe a very dramatic change to the duration profile when compared to the case with no flows. Focusing again on the case of no yield curve slope and no correlation between yield curve movements and growth assets, we notice that duration assets start with the maximum allowable duration of 30 years and then very rapidly decrease to a duration profile that lies below that of the case with no inflow and outflows. By going back to Equation (7), assuming a constant allocation w and dropping irrelevant terms and re-expanding the partial sum of the cash flows into individual cash flows, we obtain the following duration equation:

$$D_t = T - t + \frac{1}{(1-w)(C_0 + \cdots + C_t)} \times \underbrace{\sum_{k=t+1}^{T-1} (1-w) C_k (T-k)}_{\substack{\text{contribution of} \\ \text{future cash flows}}}.$$

The latter term in the above equation can be viewed as an adjustment to the previously derived result. Observing this term more closely, a clear pattern emerges. Imagine that the duration asset portion of all future inflows (or outflows) are used to buy (or sell) bonds maturing at time T. If there is a way to buy (or sell) those bonds at today's rate, then the duration assets would not contribute to any risk at the end of the investment horizon. By extending (or shortening) the duration, the investor is able to effectively lock in buying (or selling) of duration assets with future inflows (or outflows) at today's rate.

The implications of the above are profound, particularly in situations with prolonged inflows, where optimal solutions could require dramatic duration extensions. While this may sound counterintuitive, in the absence of directional views on interest rates, this will lead to more efficient long-term results.

Duration should be adjusted based on the slope of the yield curve, expected correlation of duration and growth assets, and expected yield curve changes

Panel C in Figures 3 and 4 demonstrates the sensitivity of solutions to different yield curve slopes as well as to different correlations between duration and growth asset returns. The observed patterns can also be linked to Equation (7) where we can postulate, ceteris paribus, the following points:

- If the yield curve is positively sloped, duration should be extended. This would allow investors to earn higher returns by investing in higher yielding assets and provide the opportunity to earn additional roll-down returns in more opportunistic fixed-income strategies.
- If yields are positively correlated with growth asset returns (i.e., duration asset returns are negatively correlated with growth asset returns), duration should be extended. This is in line with standard diversification benefits derived from exploiting negative correlations.
- If yields are expected to increase, duration should be shortened. This would allow investors to reinvest their assets at increasingly higher rates.

While forming expectations about future yield levels may be challenging, the more persistent nature of the slope of the yield curve and the correlation of yield curve movements with growth asset returns can present a more meaningful path through which to adjust portfolio durations (if at all).

5.2 Analytical formulation of the Income investor objective

Recall that the Income investor's objective is to maximize the probability of staying solvent throughout the investment horizon subject to a certain level of withdrawal. However, to derive the analytical heuristics, we use an analytically tractable proxy objective function which is to maximize the probability that the first-order terminal wealth is above zero. For a given amount of withdrawal to be realized in future years, we seek to maximize the objective function:

$$J(\boldsymbol{W}) = Prob(P_T^{(1)}(\boldsymbol{W}) \geq 0).$$

In Appendix C we show that maximizing the above objective is akin to choosing a solution on the mean–variance frontier obtained subject to the cash flows equal to the withdrawal amount. In particular, the solution is the point of tangency from the origin to the mean–variance frontier.

This duality principle allows us to borrow insights from the mean–variance framework and apply them to the income problem without explicitly solving for them analytically.

6 Simulation-Based Portfolio Selection (SBPS)

While the analytical framework serves the essential purpose of explaining the behavior of the growth and duration assets in the multi-period context, analytical approaches become increasingly impractical when seeking to capture higher-order effects, increasing the number of asset classes under consideration, introducing more realistic asset pricing processes, and capturing investment frictions such as transaction costs and taxes. Analytical approaches also lack the flexibility to easily consider different objective functions.

At the core of the of the SBPS approach is the optimization process that takes simulated asset returns and a relevant objective function and solves for all the weight vectors through time $(\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{T-1}) = \boldsymbol{W}$ **all at once** as opposed to a period-by-period stepwise approach. As long as the investor has a model to simulate the portfolio wealth through time for a given $\boldsymbol{W}$ in the presence of frictions, the optimization engine can solve for $\boldsymbol{W}$ in an iterative fashion using an advanced momentum-based gradient search algorithm. Additionally, such an approach allows us to optimize any objective function.

6.1 Simulation

Central to the solutions produced by SBPS are the underlying stochastic processes used to simulate the evolution of asset prices across the investment horizon. Details regarding simulation models are out of scope for this paper. We assume practitioners can employ a simulation model that can reflect real-world price dynamics. For the analyses presented in this paper, we used the Moody's Analytics Economic Scenario Generator (ESG) engine to simulate the time-series used for our analyses. This is one of the most advanced simulation engines available. It simulates nominal and real rates that are consistent with a simulated inflation process. Asset returns are then simulated in a way that ensures consistency with the underlying rate processes.

6.2 The optimization algorithm

A key innovation of the SBPS approach is that the optimization process simultaneously considers the totality of the investment horizon and allocation decisions across the investment horizon. This is in contrast to a period-by-period approach, such as recursive dynamic programming, which looks at the optimal allocation decision at time t agnostic of the decisions until time t. SBPS involves updating all of the weight vectors $(\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{T-1}) = \boldsymbol{W}$ simultaneously as the optimization progresses. We use a large-scale momentum-based gradient search algorithm introduced by Kingba and Lei Ba (2015) called *Adam* (short for Adaptive Moment Estimation) which allows us to work with a large number of weight variables and also provides the flexibility to optimize non-convex objective functions. For any given objective function $J(\boldsymbol{W})$ the optimization problem is:

$$\underset{\boldsymbol{W}=(\boldsymbol{w}_0,\boldsymbol{w}_1,\ldots,\boldsymbol{w}_{T-1})}{\text{maximize}} \; J(\boldsymbol{W}).$$

The optimization algorithm iteratively updates a sequence of weights $\boldsymbol{W}^{(0)}, \boldsymbol{W}^{(1)}, \boldsymbol{W}^{(2)}, \ldots$ until a stopping criterion is satisfied. $\boldsymbol{W}^{(0)} = (\boldsymbol{w}_0^{(0)}, \boldsymbol{w}_1^{(0)}, \ldots, \boldsymbol{w}_{T-1}^{(0)})$ is a suitably chosen initial starting weights of the algorithm.

The weight vectors are updated in the direction of the exponentially weighted moving average (EWMA) of the gradient as opposed to the current gradient. Using an EWMA of gradients addresses some of the key practical challenges which we will discuss later. The size of the update steps in the direction of the EWMA of gradient is controlled by an EWMA of squared gradients. Each iteration consists of following four steps:

***Step* 1: Compute the current gradient**

Given the weights from the last step $\boldsymbol{W}^{(n)} = (\boldsymbol{w}_0^{(n)}, \boldsymbol{w}_1^{(n)}, \ldots, \boldsymbol{w}_{T-1}^{(n)})$, the current gradient vector $\boldsymbol{g}^{(n)}$ is computed as:

$$\begin{aligned} \boldsymbol{g}^{(n)} &= \frac{\partial J(\boldsymbol{W}^{(n)})}{\partial \boldsymbol{W}^{(n)}} \\ &= \frac{\partial J(\boldsymbol{w}_0^{(n)}, \boldsymbol{w}_1^{(n)}, \ldots, \boldsymbol{w}_{T-1}^{(n)})}{\partial \boldsymbol{w}_0^{(n)} \partial \boldsymbol{w}_1^{(n)} \ldots \partial \boldsymbol{w}_{T-1}^{(n)}}. \end{aligned}$$

Standard gradient calculation approaches, such as numerically computing the gradient, involve computing the objective function as many times as the number of unknowns. For a problem involving $N = 100$ assets and $T = 40$ time periods, this involves computing the objective function $N \times T = 4{,}000$ times which is not feasible when the objective is simulation based. Consequently, our approach uses automatic differentiation to compute the gradient in a fast and efficient manner. In computer algebra, every function can be viewed as a series of arithmetic operations. Automatic differentiation leverages this idea and expresses the gradient of the function as a chain of differentials corresponding to each arithmetic operation. It utilizes the fact that the differentials corresponding to each arithmetic operation are simple and can be expressed analytically, and further, by chaining them together according to the chain rule, automatic differentiation is able to provide a faster method of gradient calculation.

***Step* 2: Update EWMA of the gradient**

The EWMA of gradients $\boldsymbol{g}^{(0)}, \boldsymbol{g}^{(1)}, \ldots$ is the sequence of vectors $\boldsymbol{m}^{(0)}, \boldsymbol{m}^{(1)}, \ldots$ such that $\boldsymbol{m}^{(0)} = \mathbf{0}$ and

$$\boldsymbol{m}^{(n+1)} = \beta_1 \boldsymbol{m}^{(n)} + (1 - \beta_1)\boldsymbol{g}^{(n)},$$

where $0 < \beta_1 < 1$ is an exponential decay parameter. As we can see the EWMA gradient is biased towards the starting $\boldsymbol{m}^{(0)}$ which is $\mathbf{0}$. If not bias corrected, this requires a large number of iterations to mitigate the bias when β_1 is close to 1. Adam corrects this behavior by using the bias corrected EWMA gradient:

$$\hat{m}^{(n+1)} = \frac{\boldsymbol{m}^{(n+1)}}{1 - \beta_1^{n+1}}.$$

***Step* 3: Update EWMA of the squared gradient**

The EWMA of squared gradients $\boldsymbol{g}^{(0)2}, \boldsymbol{g}^{(1)2}, \ldots$ is the sequence of vectors $\boldsymbol{v}^{(0)}, \boldsymbol{v}^{(1)}, \ldots$ such that $\boldsymbol{v}^{(0)} = \mathbf{0}$ and

$$\boldsymbol{v}^{(n+1)} = \beta_2 \boldsymbol{v}^{(n)} + (1 - \beta_2)\boldsymbol{g}^{(n)2},$$

where $0 < \beta_2 < 1$ is an exponential decay parameter. Similar to the previous step, the bias corrected EWMA of squared gradient is:

$$\hat{v}^{(n+1)} = \frac{\boldsymbol{v}^{(n+1)}}{1 - \beta_2^{n+1}}.$$

***Step* 4: Update weights**

Once the EWMA of the gradient and EWMA of the squared gradients are obtained, the updated weights are:

$$\begin{aligned} \boldsymbol{W}^{(n+1)} &= \left(\boldsymbol{w}_0^{(n+1)}, \boldsymbol{w}_1^{(n+1)}, \ldots, \boldsymbol{w}_{T-1}^{(n+1)}\right) \\ &= \left(\boldsymbol{w}_0^{(n)}, \boldsymbol{w}_1^{(n)}, \ldots, \boldsymbol{w}_{T-1}^{(n)}\right) \\ &\quad + h \frac{\hat{m}^{(n+1)}}{\sqrt{\hat{v}^{(n+1)}} + \varepsilon}, \end{aligned}$$

where h is a parameter controlling the update step size and is referred to as the learning rate. To avoid division by zero errors the algorithm uses $\sqrt{\hat{v}^{(n)}} + \varepsilon$ in the denominator as opposed to $\sqrt{\hat{v}^{(n)}}$, where ε is a small positive number typically of the order of 10^{-6}. Ignoring ε, the ratio $\hat{m}^{(n)}/\sqrt{\hat{v}^{(n)}}$ is called the *signal-to-noise* ratio (SNR). The algorithm slows down in noisy parts of the surface as the SNR decreases, conversely, the algorithm speeds up while passing through less noisy parts of the surface where the SNR increases.

End: *Repeat steps* 1 *through* 4 *until convergence*

The use of the EWMA of gradients and squared gradients allows Adam to address a number of challenges including:

- Surfaces with local maxima where the gradient evaluates to zero and the standard gradient search algorithm stops
- Situations where the gradient search has trouble navigating in areas where the gradient changes much more quickly in one dimension than in others
- Noisy surfaces (such as simulation-based objectives) where the gradient can be unstable

We note that the optimization algorithm applies 50 basis points (bps) transaction costs for the simulation-based solutions we present.

We conclude this section by pointing out that advances in computing power have only recently allowed for solving the multi-period portfolio selection problem using a simulation-based approach.[6]

6.3 Analytical versus simulation-based results

Before putting SBPS to use, we first ran a series of tests to confirm that the SBPS approach produces results consistent with the analytical principles outlined in the previous section. To do this, we first simulated assets using the same pricing dynamics outlined in the analytical section and then compared the simulation-based results of a *Median vs Median* -15*th percentile* objective to the analytical *Mean vs Variance* results.[7] The comparison produced favorable results with respect to all principles and showed that allocations and duration profiles produced using SBPS were well aligned with intuitive analytical expectations. Details of this exercise are provided in Appendix D.

6.4 SBPS examples and key insights

So far, we have presented cases with limited investment opportunity sets to allow for a simplified exposition of expected allocation dynamics over time. We now move to presenting cases with much broader investment opportunity sets that are more relevant to investors from a practical perspective.[8] For these cases we simulate 100,000 paths of annualized total return (TR) time-series over the investment horizons considered for the following set of assets:

- US equities
- Developed ex US equities
- Emerging market equities
- US Treasury zero coupon bonds (1-year, 2-year, ..., 30-year constant maturity TR indexes)
- TIPS (1-year, 2-year, ..., 30-year constant maturity TR indexes)

A subset of summary statistics of the simulated data is presented in Appendix F.

The use of a set of constant maturity indexes allows us to replicate the performance of a buy and hold zero-coupon bonds (or TIPS bonds). For example, to replicate the performance of a buy and hold T-maturity zero-coupon bond (or TIPS bond), one starts with the T-maturity zero-coupon index (or TIPS index) at time $t = 0$. At time $t = 1$, the T-maturity zero-coupon index (or TIPS index) is traded for a $T - 1$-maturity zero-coupon index (or TIPS index) and so on. Finally, at time $t = T - 1$, the 2-year maturity zero-coupon (or TIPS) index is traded for a 1-year maturity zero-coupon (or TIPS) index. The optimizer is therefore able to replicate hold-to-maturity securities from the above indexes.

For all SBPS examples that follow, it should be noted that we will be presenting unconstrained solutions so as to provide the clearest and most informative exposition of the SBPS approach. For real-world applications, investors will likely impose practical constraints on one or more assets.

6.4.1 SBPS for the growth investor

Our first set of examples of the practical application of SBPS will be based on the two Growth investor scenarios presented in the analytical section. We produce simulation-based Median vs Median–5th percentile efficient frontiers by maximizing Median–λ (Median – 5th percentile) for different levels of the parameter $\lambda \geq 0$. We highlight three solution points on each of the simulation-based efficient frontiers, namely a low-risk solution, a medium-risk solution, and a high-risk solution. Cash flows, simulation-based

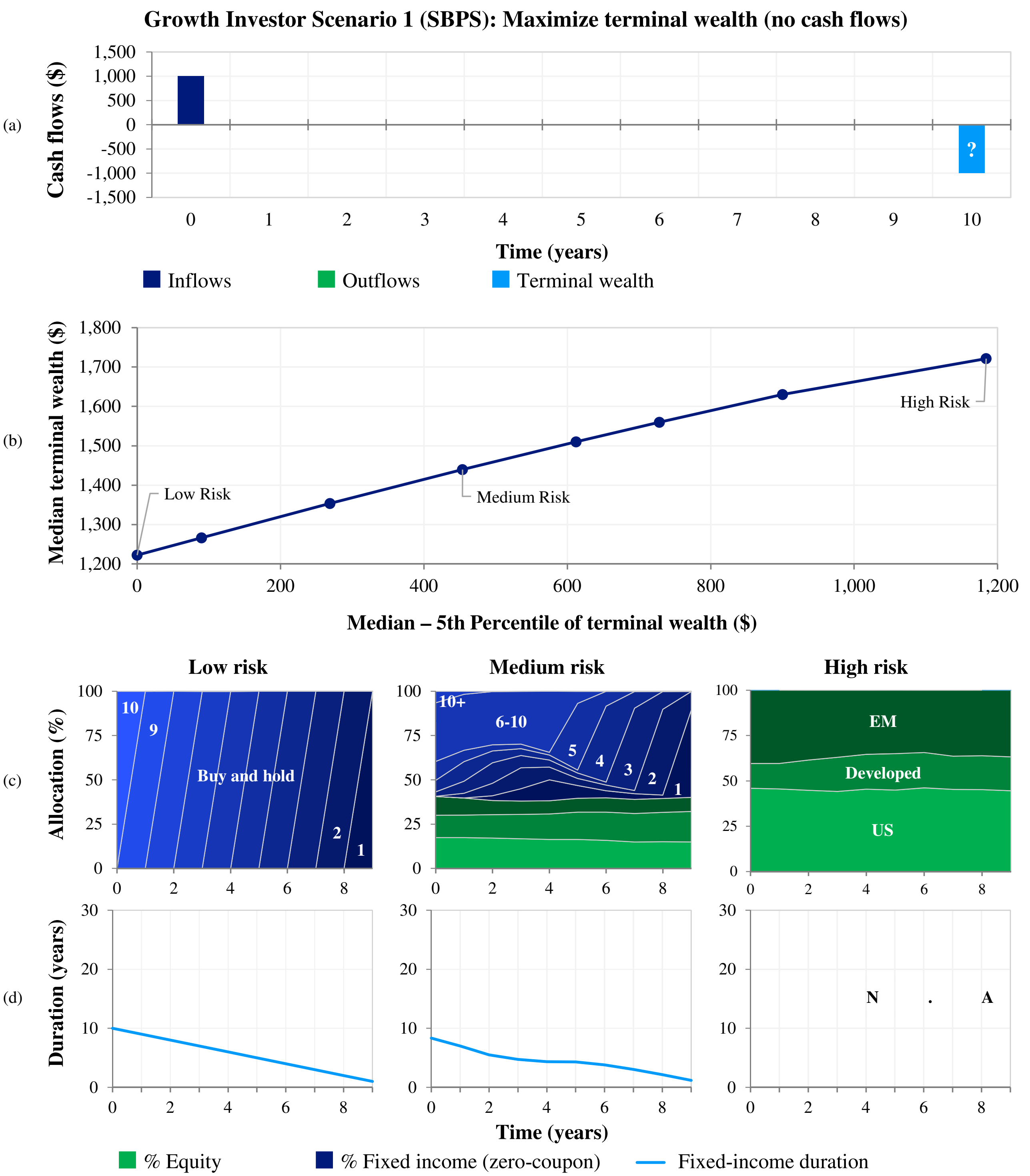

Figure 5 **(a)** Scenario cash inflows/outflows. Terminal wealth at $t = 10$ to be determined. **(b)** Efficient Frontier of median wealth vs. median – 5th percentile terminal wealth. **(c)** Optimal allocations and **(d)** fixed-income durations for selected solutions.

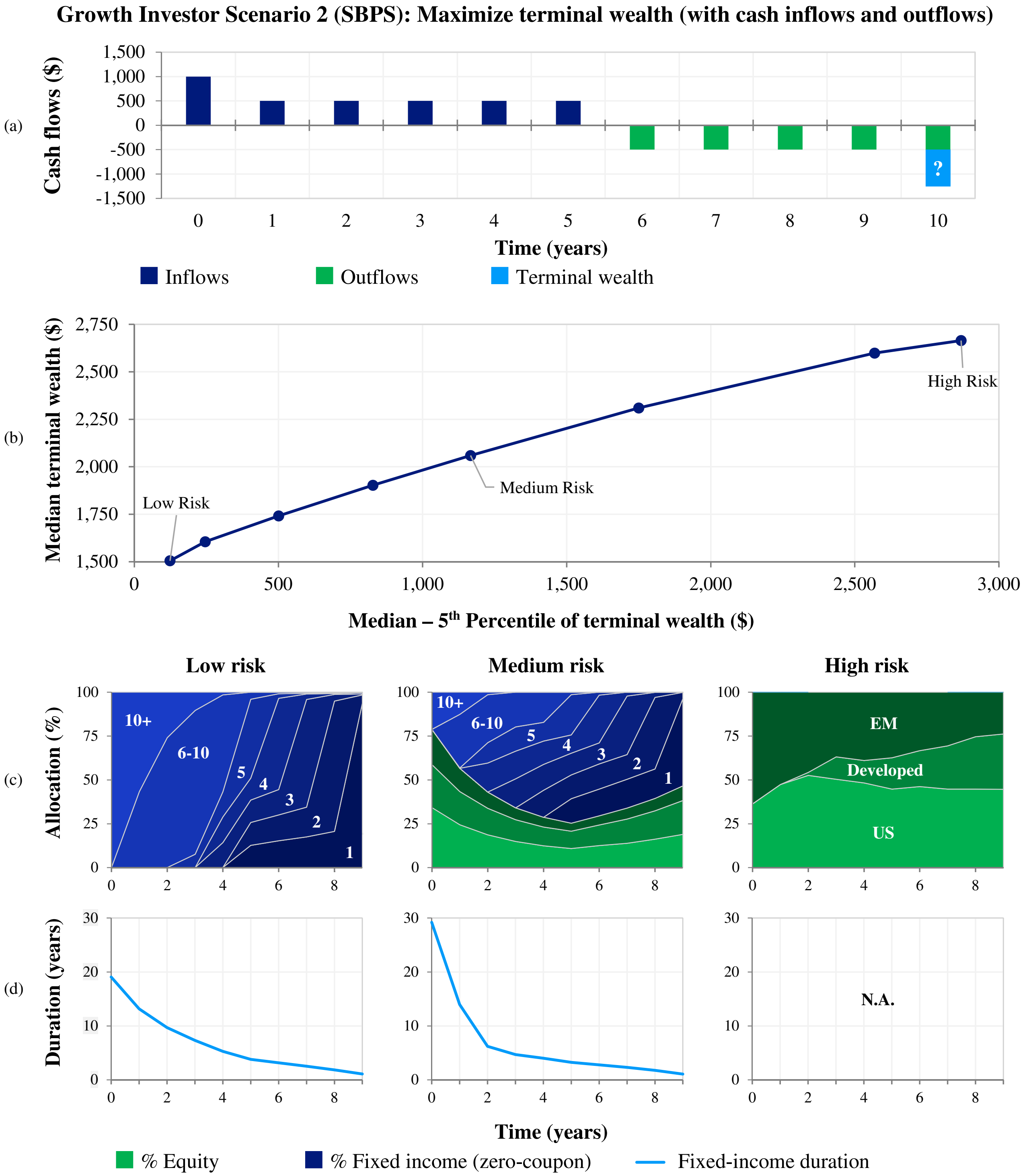

Figure 6 **(a)** Scenario cash inflows/outflows. Terminal wealth at $t = 10$ to be determined **(b)** Efficient Frontier of median wealth vs. median – 5th percentile terminal wealth. **(c)** Optimal allocations and **(d)** fixed-income durations for selected solutions.

efficient frontiers, allocations for selected solutions, and durations for selected solutions based on Growth investor scenarios 1 and 2 are shown in Figures 5 and 6, respectively.

Based on the SBPS results we make the following key observations in the context of an investor that is looking to optimize his future wealth:

SBPS generates allocations and duration profiles aligned with analytical expectations

The results show that all the key principles derived from the analytical examples still hold true. Specifically, we observe the following:

- Allocations shift to safer (riskier) allocations as inflows (outflows) occur
- Duration profiles align with investment horizons when there are no cash flows
- Duration is extended (shortened) in the presence of future inflows (outflows)

SBPS allows for long-term real-world asset dynamics

The simulation-based approach allows us to model assets in more flexible and realistic ways that can be incorporated into a simplistic analytical model. For example, all fixed-income instruments are priced off of simulated interest rate curves. This allows for such instruments to be more consistently priced while also better reflecting current market realities (i.e., meaningfully skewed due to a low rate environment). This means that interest rate curve paths are in turn aligned with market realities (i.e., meaningfully skewed due to a low rate environment). Correlations between rates and equities are assumed to be slightly negative (as opposed to the slightly positive correlation experience over the past decade).

Effects like the ones mentioned above are directly incorporated and meaningfully impact the resulting solutions. One such example can be seen in the medium-risk solution for Scenario 1 (no cash flows) where we see a shortening of duration early on to enhance returns (center chart of panel C in Figure 5). This is due to the simulated set of interest rate paths capturing a changing interest rate environment. Observe in Table F1 of Appendix F, the shorter maturity zero-coupon bonds, namely 1-year to 5-year, exhibit higher returns than longer maturity zero-coupon bonds in the earlier years reflecting an embedded rising rate expectation. In this case, a balance is being struck between matching duration and seeking returns. The assumption of slightly negative correlation between rates and equities further magnifies the observed behavior.

SBPS allows for the consideration of investment frictions and illiquidities

Simulation also allows for consideration of investment frictions and illiquidities. The simulations used for these analyses include assumed transaction costs of 50 bps. This not only incorporates the impact of transaction costs on expected outcomes, but also encourages the optimization process to produce solutions that exhibit lower turnover. An example of this can again be seen in Scenario 1 (no cash flows) where the lowest risk solution (left-most chart of panel C in Figure 5) involves holding a sequence of 10-year, 9-year, . . . , 1-year zero-coupon indexes through time. This sequence of holdings is partially driven by the avoidance of transaction costs. This point is better understood when one realizes that this sequence of holdings is equivalent to buying and holding a 10-year Treasury zero-coupon bond to maturity.

6.4.2 SBPS for the income investor

Our second example of the practical application of SBPS is that of an income investor. Here we consider an investor who wants to withdraw a fixed amount of income in retirement. The investor is currently 40 years old and plans to retire at age 60. He expects to pull annual consumption cash flows from his portfolio for 20 years.

Income Investor: Maximize real consumption cash flows for a given probability of remaining solvent.

- A 40-year investment horizon
- An initial investment of \$100,000
- Annual inflows of \$50,000 (nominal) for the first 20 years
- Annual outflows begin in year 20 and continue until year 40

Note that we allow for the possibility of working with both nominal cash flows (contributions) and real cash flows (income withdrawals). In real dollar terms, nominal contributions have less value as we go out in the future. In Appendix E, we detail the conversion of nominal dollars into real dollars and vice versa. This conversion allows us to work in one single unit.

The goal of the income maximization problem is to maximize the probability of success for a given level of income withdrawal. From a risk–reward perspective, the income levels in the y-axis represent our reward and one minus the probability of success on the x-axis is our measure of risk. We maximize the objective $J(\boldsymbol{W}) = Prob(\tau_A(\boldsymbol{W}) \geq T + 1)$ which is the probability of staying solvent until the end of the investment horizon for the income withdrawal amount denoted by the parameter A. We produce the Income investor's efficient frontier by increasing the income level A and plotting it against one minus the maximized probability of success. In Figure 7 we present the cash flows in real dollar terms, the income investor's efficient frontier, and the allocations and the real durations for the three selected solutions. Here we note that the highest risk solution on the income frontier has 100% allocated to riskier assets during inflows. On increasing the income level further, the allocation does not change materially, and the exercise reduces to simply simulating the probability of success for increasing levels of withdrawal as opposed to running an optimization. This part of the income frontier is represented by the dotted portion in panel (b) of Figure 7.

Lowest risk solutions seek duration early on and diversify with growth assets in later years

Observing the lowest risk allocation in panel C of Figure 7, the solution focuses on duration assets with extended durations early in the investment horizon. At this early stage, it is most important to eliminate the interest rate risk associated with future purchases (and dispositions) of bonds that will be required based on expected upcoming inflows/outflows. As the portfolio progresses through the investment horizon and expected outflows increasingly outweigh expected inflows, solutions require less dollar duration. Consequently, allocations gradually shift to shorter duration bonds and introduce growth asset exposures for diversification purposes.

Highest risk solutions focus on growth assets early with duration being added in later years

In the high-risk solution in panel C of Figure 7, we see that instead of maintaining a 100% allocation to growth assets through time, the solution moves to safer assets as outflows begin. Intuitively, during inflows, the solution seeks maximum return and, consequently, maximum risk. The outflow period is not as intuitive but can be explained as follows: Once outflows begin, they are assumed to be constant and there is also no utility for any remaining portfolio balance following those outflows. As a result, for a given cash outflow amount, the more successful paths will have no remaining upside and will only be exposed to the downside risk of becoming insolvent. To mitigate

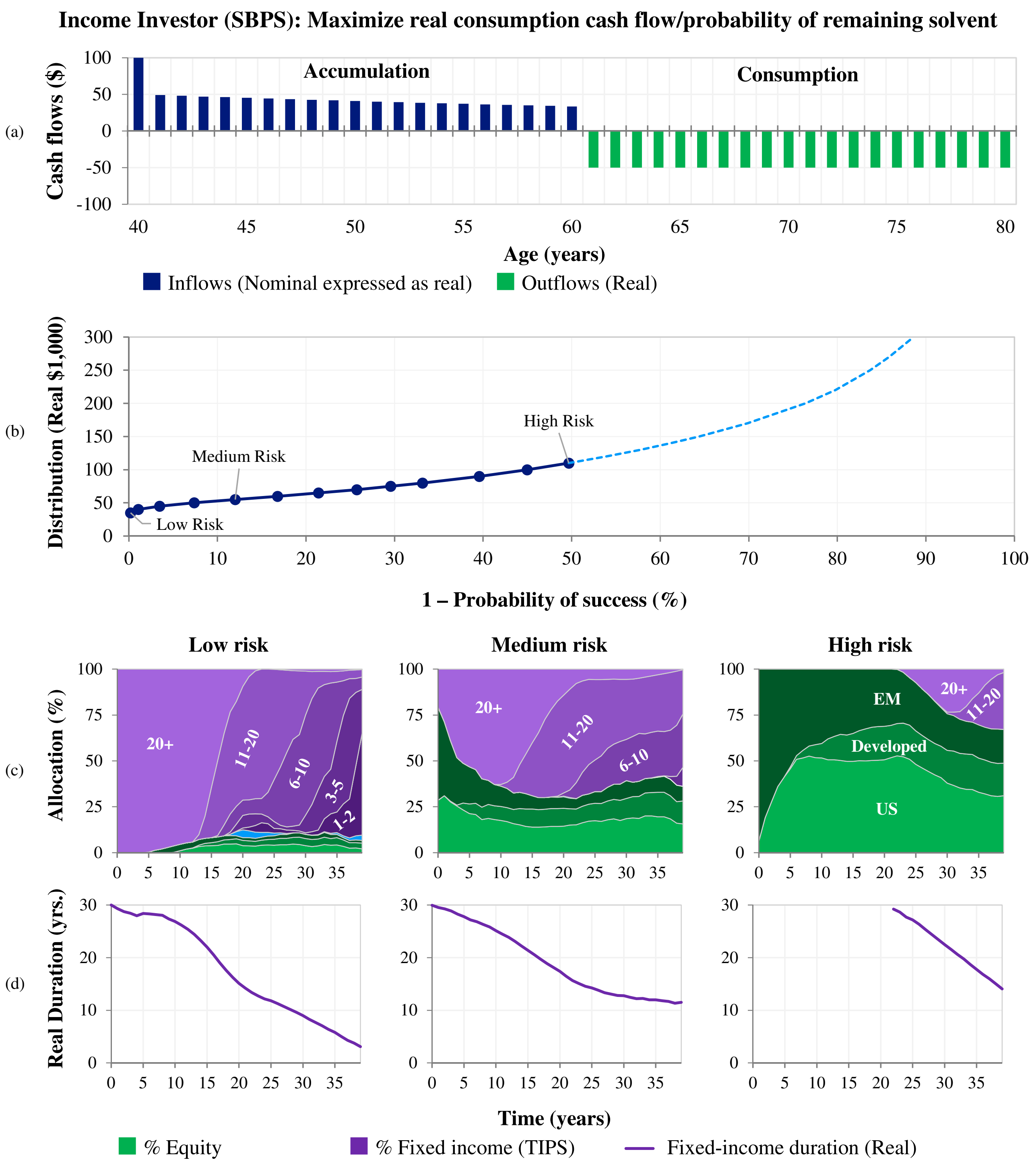

Figure 7 **(a)** Cash inflows/outflows in real dollar terms. **(b)** Efficient frontier of distribution vs. 1-probability of success. The dotted line represents the simulated probabilities beyond the highest risk solution. **(c)** Optimal allocations and **(d)** fixed-income durations of selected solutions.

this effect, once the outflows begin, the portfolio is seen to gradually move to some duration assets.

7 Conclusion

In this paper, we presented a simulation-based portfolio selection framework that addresses what we initially established as three requisites for the development of practical multi-period solutions:

1. Solutions must evolve allocations and duration over time to align with expected cash flows
2. Solutions must consider real-world asset dynamics
3. Solutions must consider investment frictions and illiquidities

To provide intuition for the multi-period problem, we detailed an analytical framework that considers the distinction between growth and duration assets. As part of this process we also defined objective functions for two common investor types: Growth investors, who seek to maximize terminal wealth, and Income investors, who seek to maximize a series of future cash flows. We presented analytical solutions for these objectives under scenarios where no cash flows were expected and where both cash inflows and outflows were expected and under cases when only growth assets were considered and when both growth and duration assets were considered. We then analyzed sensitivities of allocations and durations to cash flows, correlations between assets and to the slope of the yield curve. Through this analysis we developed several guiding principles relating to how portfolio allocations and durations should evolve over a multi-period investment horizon.

Finally, we introduced a simulation-based portfolio selection framework that decomposes the multi-period problem into three distinct parts: the objective function, simulation, and optimization. Using the growth investor and income investor objective functions, asset simulations that incorporated real-world asset dynamics, and an optimization algorithm that simultaneously considers the totality of the investment horizon and allocation decisions across the investment horizon, we then produced multi-period solutions that were not only aligned with the principles developed as part of our analytical exercise but also demonstrated the flexibility afforded by SBPS in allowing us to more effectively address the three stated requisites for practical multi-period solutions.

Key innovations of this research include:

1. The development of an analytical framework for the multi-period problem that provides a theoretical foundation for multi-period portfolio selection and provides intuition for how portfolio allocations and duration should evolve through time.
2. The development of a flexible simulation-based multi-period portfolio selection framework that addresses the practical realities of implementing and managing multi-period solutions and allows for the incorporation of individual hold-to-maturity fixed-income investments alongside traditional investments used for strategic asset allocation.
3. Methods for leveraging of advances in computing power and machine learning algorithms to solve multi-period portfolio selection problems using a simulation-based approach.

Acknowledgments

We are grateful to Mark Kritzman and Marc Reinganum for their helpful comments and discussions.

Appendix A: Analytical Framework for the Growth Investor'S Objective Without Duration Assets

The $N \times 1$ vectors $\boldsymbol{r}_t$ are the return vectors at $t = 0, 1, \ldots, T-1$ and they are assumed to be independently and identically distributed (i.i.d.) normal random variables with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The $N \times 1$ weight vectors $\boldsymbol{w}_t$ at $t = 0, 1, \ldots, T-1$ can be thought of having $N-1$ free weights in the first $N-1$ assets represented by $\boldsymbol{w}_t^{(N-1)} = (w_{t,1}, w_{t,2}, \ldots, w_{t,N-1})$ and the weight of the N-th asset is $w_{t,N} = 1 - \mathbf{1}^T \boldsymbol{w}_t^{(N-1)}$ where $\mathbf{1}$ is the $N-1 \times 1$ vector of all ones. We also introduce the $N \times 1$ index vector $\boldsymbol{e}_k$ which has one at the k-th position and zero everywhere else; and let $\boldsymbol{I}$ be the $N-1 \times N-1$ identity matrix. The weight vector becomes $\boldsymbol{w}_t = \boldsymbol{e}_N + (\boldsymbol{I} - \mathbf{1})^T \boldsymbol{w}_t^{(N-1)}$. Recall that the first-order approximation of terminal wealth from Equation (4) is:

$$P_T^{(1)}(W) = \sum_{t=0}^{T-1} X_t \boldsymbol{w}_t^T \boldsymbol{r}_t + X_T.$$

Therefore, the mean and variance of the terminal wealth is:

$$E(P_T^{(1)}) = X_T + \sum_{t=0}^{T-1} X_t \boldsymbol{w}_t^T \boldsymbol{\mu} \quad \text{and}$$

$$V(P_T^{(1)}) = \sum_{t=0}^{T-1} X_t^2 \boldsymbol{w}_t^T \boldsymbol{\Sigma} \boldsymbol{w}_t.$$

The covariance term in the variance formula disappears since we assume that return vectors are independent across time. The partial derivatives of the expectation and variance with respect to $\boldsymbol{w}_t^{(N-1)}$ are:

$$\frac{\partial E(P_T^{(1)})}{\partial \boldsymbol{w}_t^{(N-1)}} = X_t (\boldsymbol{I}\ -\mathbf{1}) \boldsymbol{\mu} = X_t \boldsymbol{m}$$

and

$$\begin{aligned}\frac{\partial V(P_T^{(1)})}{\partial \boldsymbol{w}_t^{(N-1)}} &= 2X_t^2 \frac{\partial \boldsymbol{w}_t}{\partial \boldsymbol{w}_t^{(N-1)}} \Sigma \boldsymbol{w}_t \\ &= 2X_t^2 (\boldsymbol{I}\ -\mathbf{1}) \boldsymbol{\Sigma} \boldsymbol{w}_t \\ &= 2X_t^2 (\boldsymbol{I}\ -\mathbf{1})\ \boldsymbol{\Sigma} \boldsymbol{e}_N + 2X_t^2 (\boldsymbol{I}\ -\mathbf{1}) \\ &\quad \times \boldsymbol{\Sigma} (\boldsymbol{I} - \mathbf{1})^T \boldsymbol{w}_t^{(N-1)} \\ &= -2X_t^2 \boldsymbol{b} + 2X_t^2 \boldsymbol{A} \boldsymbol{w}_t^{(N-1)},\end{aligned}$$

where

$$\boldsymbol{m} = (\boldsymbol{I}\ -\mathbf{1}) \boldsymbol{\mu}$$

$$\boldsymbol{b} = (-\boldsymbol{I}\ \mathbf{1}) \boldsymbol{\Sigma} \boldsymbol{e}_N$$

$$\boldsymbol{A} = (\boldsymbol{I}\ -\mathbf{1}) \boldsymbol{\Sigma} (\boldsymbol{I}\ -\mathbf{1})^T.$$

We note that if $\boldsymbol{\Sigma}$ is positive definite, then so is $\boldsymbol{A}$; this is because if $\boldsymbol{c}$ is any non-zero $N-1 \times 1$ vector, then $\boldsymbol{d} = (\boldsymbol{I}\ -\mathbf{1})^T \boldsymbol{c}$ is also non-zero which implies $\boldsymbol{c}^{\boldsymbol{T}} \boldsymbol{A} c = \boldsymbol{d}^{\boldsymbol{T}} \boldsymbol{\Sigma} d > 0$. The positive definiteness of $\boldsymbol{A}$ ensures that by equating $\partial E(P_T^{(1)})/\partial \boldsymbol{w}_t^{(N-1)} - \lambda \partial V(P_T^{(1)})/\partial \boldsymbol{w}_t^{(N-1)}$ to $\mathbf{0}$ we can obtain a unique maximizing solution to the mean–variance problem, and the solution is given by:

$$\boldsymbol{w}_t^{(N-1)} = \boldsymbol{A}^{-1} \left(\frac{1}{2\lambda X_t} \boldsymbol{m} + \boldsymbol{b} \right).$$

Appendix B: Analytical Framework for the Growth Investor's Objective with Duration Assets

Consider portfolio selection with one growth asset and an arbitrary collection of duration assets. At any given time t the portfolio has w_t invested in the growth asset and $1 - w_t$ invested in duration assets with duration D_t. The duration asset at time $t+1$ has a remaining duration of $D_t - 1$. Under our simplistic yield process, the yield curve has a total slope of s up to maturity $T_{\max}$, i.e., the slope increases by $\Delta s = s/T_{\max}$ for each maturity point. The yield of the duration asset at time t and

$t+1$ are $y_t + \Delta s D_t$ and $y_{t+1} + \Delta s(D_t - 1)$ respectively. We have used the assumption that the yield of this bond is roughly equal to the yield of a D_t-maturity zero coupon bond. The one period return of the duration asset is given by:

$$
\begin{aligned}
r_t^{(D_t)} &= \frac{\exp(-(D_t-1)(y_{t+1} + \Delta s(D_t - 1)))}{\exp(-D_t(y_t + \Delta s D_t))} - 1 \\
&= \exp(-D_t \Delta y_t + 2\Delta s D_t - \Delta s + y_t \\
&\quad + \Delta y_t) - 1 \\
&\approx y_t - \Delta y_t(D_t - 1) + (2D_t - 1)\Delta s \\
&= \underbrace{yt + \Delta s D_t}_{\textit{yield return}} + \underbrace{\Delta s(D_t - 1)}_{\substack{\textit{carry from rolling} \\ \textit{down the yield curve}}} \\
&\quad - \underbrace{\Delta y_t(D_t - 1)}_{\textit{random}}.
\end{aligned}
$$

Using $y_t = y_0 + \Delta y_0 + \Delta y_1 + \cdots + \Delta y_{t-1}$ we can also write $r_t^{(D_t)}$ as:

$$
\begin{aligned}
r_t^{(D_t)} &= (y_0 + 2\Delta s D_t - \Delta s) + (\Delta y_0 + \Delta y_1 \\
&\quad + \cdots + \Delta y_t - D_t \Delta y_t).
\end{aligned}
$$

Let $\boldsymbol{\mu_D}$ be the vector of the mean returns of the duration asset through time. Then, the t-th element of $\boldsymbol{\mu_D}$ is given by $E(r_t^{(D_t)}) = (y_0 + 2\Delta s D_t - \Delta s) + (\mu_{y,0} + \mu_{y,1} + \cdots + \mu_{y,t} - \mu_{y,t} D_t)$. The first-order terminal wealth can be re-written as:

$$
\begin{aligned}
P_T^{(1)} &= X_T + \mu_g \boldsymbol{w}^T \boldsymbol{X_d} \mathbf{1} + (\mathbf{1} - \boldsymbol{w})^T \boldsymbol{X_d} \boldsymbol{\mu_D} \\
&\quad + \sum_{t=0}^{T-1} X_t \sigma_g z_{g,t} \\
&\quad + \sum_{t=0}^{T-1} \sigma_y z_{y,t} h_t(w_t, \ldots, w_{T-1}, D_t),
\end{aligned}
$$

where the equity returns are expressed as $r_t^{(E)} = \mu_g + \sigma_g z_{g,t}$ and the yield changes as $\Delta y_t = \mu_{y,t} + \sigma_y z_{y,t}$. The random variables $z_{g,t}$ and $z_{y,t}$ are $N(0, 1)$ such that $\mathrm{cov}(z_{g,t}, z_{y,t}) = \rho_{g,y}$ and $h_t(w_t, \ldots, w_{T-1}, D_t) = X_t(1 - w_t) + \cdots + X_{T-1}(1 - w_{T-1}) - X_t(1 - w_t)D_t$.

The expectation and the variance of the terminal wealth are given by:

$$
E(P_T^{(1)}) = X_T + \mu_g \boldsymbol{w}^T \boldsymbol{X_d} \mathbf{1} + (\mathbf{1} - \boldsymbol{w})^T \boldsymbol{X_d} \boldsymbol{\mu}_D
$$

and

$$
\begin{aligned}
V(P_T^{(1)}) &= \sigma_g^2 \boldsymbol{w}^T \boldsymbol{X_d}^2 \boldsymbol{w} + \sigma_y^2 (\mathbf{1} - \boldsymbol{w})^T \boldsymbol{X_u}^T \boldsymbol{X_u} (\mathbf{1} - \boldsymbol{w}) \\
&\quad + 2\rho_{g,y} \sigma_g \sigma_y \boldsymbol{w}^T \boldsymbol{X_d} \boldsymbol{X_u} (\mathbf{1} - \mathbf{w}),
\end{aligned}
$$

where

$$
\boldsymbol{\mu_D} = \begin{pmatrix}
y_0 + 2\Delta s D_0 - \Delta s \\
\quad + \mu_{y,0} - \mu_{y,0} D_0 \\
y_0 + 2\Delta s D_1 - \Delta s \\
\quad + \mu_{y,0} + \mu_{y,1} - \mu_{y,1} D_1 \\
\vdots \\
y_0 + 2\Delta s D_{T-1} - \Delta s \\
\quad + \mu_{y,0} + \cdots \\
\quad + \mu_{y,T-1} - \mu_{y,T-1} D_{T-1}
\end{pmatrix}
$$

$$
\boldsymbol{X_d} = \begin{pmatrix}
X_0 & 0 & \cdots & 0 \\
0 & X_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X_{T-1}
\end{pmatrix} \text{ and}
$$

$$
\boldsymbol{X_u} = \begin{pmatrix}
X_0 & X_1 & \cdots & X_{T-1} \\
0 & X_1 & \cdots & X_{T-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X_{T-1}
\end{pmatrix} - \mathrm{diag}\,(\boldsymbol{D}) \boldsymbol{X}_d.
$$

B.1 *Weight formula*

The weight formula can be obtained analytically if the optimal durations to hold through time are known. If the optimal durations are known, the objective $E(P_T^{(1)}) - \lambda V(P_T^{(1)})$ is quadratic in

$\boldsymbol{w} = (w_0, w_1, \ldots, w_{T-1})$. The partial derivatives of the expectation and variance of the first-order terminal wealth are:

$$\frac{\partial E(P_T^{(1)})}{\partial \boldsymbol{w}} = \mu_g \boldsymbol{X_d} \boldsymbol{1} - \boldsymbol{X_d} \boldsymbol{\mu_D} = \boldsymbol{m}$$

and

$$\begin{aligned}\frac{\partial V(P_T^{(1)})}{\partial \boldsymbol{w}} &= 2(\sigma_g^2 \boldsymbol{X_d^2} + \sigma_y^2 \boldsymbol{X_u^T X_u} \\ &\quad - 2\rho_{g,y}\sigma_g\sigma_y \boldsymbol{X_d X_u})\boldsymbol{w} \\ &\quad + 2(\rho_{g,y}\sigma_g\sigma_y \boldsymbol{X_d X_u} \\ &\quad - \sigma_y^2 \boldsymbol{X_u^T X_u})\boldsymbol{1} \\ &= 2Uw + 2\boldsymbol{v}.\end{aligned}$$

Setting the partial derivative of the objective to zero, we obtain the optimal equity weights w.

$$\boldsymbol{w} = \boldsymbol{U}^{-1}\left(\frac{1}{2\lambda}\boldsymbol{m} - \boldsymbol{v}\right).$$

Remark (*Non-negative definiteness of* $\boldsymbol{U}$): *$\boldsymbol{U}$ must be a non-negative definite matrix so that the weight vector derived above is the maximizing solution. We show below that $\boldsymbol{U}$ is non-negative definite. Note that we can re-write $\boldsymbol{U}$ as*:

$$\begin{aligned}\boldsymbol{U} &= (\sigma_g \boldsymbol{X_d} + \sigma_y \boldsymbol{X_u})^T(\sigma_g \boldsymbol{X_d} + \sigma_y \boldsymbol{X_u}) \\ &\quad - 2(1+\rho_{g,y})\sigma_g\sigma_y \boldsymbol{X_d X_u}.\end{aligned}$$

For any real matrix $\boldsymbol{M}$, $\boldsymbol{M}^T\boldsymbol{M}$ is non-negative definite. Thus, $(\sigma_e \boldsymbol{X_d} + \sigma_y \boldsymbol{X_u})^T(\sigma_e \boldsymbol{X_d} + \sigma_y \boldsymbol{X_u})$ is non-negative definite. Additionally, when $D_t \geq 1$ for all $t = 0, 1, \ldots, T-1$, all the diagonal elements of $-\boldsymbol{X_d X_u}$ are either positive or zero making $-\boldsymbol{X_d X_u}$ a non-negative definite matrix. The sum of two non-negative definite matrixes is also non-negative definite which makes $\boldsymbol{U}$ non-negative definite. When $\boldsymbol{U}$ is not positive definite, uniqueness of the solution is not guaranteed. We can ensure uniqueness by using a regularized matrix $\boldsymbol{U} + \varepsilon\boldsymbol{I}$ which is positive definite for any $\varepsilon > 0$; typically, ε is a small number.

B.2 Duration formula

The optimal duration can be obtained analytically if the optimal weights are known. The partial derivatives of the expectation and variance with respect to D_t are:

$$\frac{\partial E(P_T^{(1)})}{\partial D_t} = (2\Delta s - \mu_{y,t})X_t(1-w_t)$$

and

$$\frac{\partial V(P_T^{(1)})}{\partial D_t} = 2\sigma_y^2 h_t \frac{\partial h_t}{\partial D_t} + 2\rho_{g,y}\sigma_g\sigma_y X_t w_t \frac{\partial h_t}{\partial D_t}.$$

Observe that $\partial h_t/\partial D_t = -X_t(1-w_t)$. Setting the partial derivative of the objective function to zero and going through a series of algebraic manipulations, we obtain the formula for D_t. The steps are:

$$\begin{aligned}&(2\Delta s - \mu_{y,t})X_t(1-w_t) + 2\lambda\sigma_y^2 h_t X_t(1-w_t) \\ &\quad + 2\lambda\rho_{g,y}\sigma_g\sigma_y X_t^2 w_t(1-w_t) = 0.\end{aligned}$$

Without loss of generality, we can assume $w_t < 1$, i.e., there is a positive allocation to the duration asset. Thus, we can simplify the above equation as:

$$\begin{aligned}2\Delta s - \mu_{y,t} + 2\lambda\sigma_y^2 h_t + 2\lambda\rho_{g,y}\sigma_g\sigma_y X_t w_t &= 0 \\ h_t + \frac{\rho_{g,y}\sigma_g X_t w_t}{\sigma_y} + \frac{2\Delta s - \mu_{y,t}}{2\lambda\sigma_y^2} &= 0.\end{aligned}$$

Using $h_t(w_t, \ldots, w_{T-1}, D_t) = X_t(1-w_t) + \cdots + X_{T-1}(1-w_{T-1}) - X_t(1-w_t)D_t$ we get

$$\begin{aligned}&X_t(1-w_t)D_t \\ &\quad = X_t(1-w_t) + \cdots + X_{T-1}(1-w_{T-1}) \\ &\quad + \frac{\rho_{g,y}\sigma_g X_t w_t}{\sigma_y} + \frac{2\Delta s - \mu_{y,t}}{2\lambda\sigma_y^2},\end{aligned}$$

which gives:

$$D_t = \sum_{k=0}^{T-1-t} \frac{X_{t+k}(1-w_{t+k})}{X_t(1-w_t)} + \frac{\rho_{g,y}\sigma_g X_t w_t}{\sigma_y X_t(1-w_t)} + \frac{2\Delta s - \mu_{y,t}}{2\lambda\sigma_y^2 X_t(1-w_t)}.$$

To observe that this is indeed the maximizing solution, we look at the second-order partial derivatives:

$$\frac{\partial^2 E(P_T^{(1)})}{\partial D_t^2} = 0 \quad \text{and}$$

$$\frac{\partial^2 V(P_T^{(1)})}{\partial D_t^2} = 2\sigma_y^2 X_t^2(1-w_t)^2,$$

which gives $\partial^2 E(P_T^{(1)})/\partial D_t^2 - \lambda\partial^2 V(P_T^{(1)})/\partial D_t^2 \leq 0$ meaning that the above solution is indeed the maximizing solution.

Remark (*Lowest Risk Solution*): *We seek a solution, if exists, such that* $V(P_T^{(1)}) = 0$, *i.e., we seek* $\boldsymbol{w} = (w_0, w_1, \ldots, w_{T-1})$ *and* $\boldsymbol{D} = (D_0, D_1, \ldots, D_{T-1})$ *that satisfies* $V(P_T^{(1)}) = 0$. *It is easy to see that* $V(P_T^{(1)}) = 0$ *when* $\boldsymbol{w} = \boldsymbol{0}$ *and* $\boldsymbol{X}_u\boldsymbol{1} = 0$. *The equality* $\boldsymbol{X}_u\boldsymbol{1} = 0$ *implies that for any* $t = 0, 1, \ldots, T-1$

$$X_t(1-D_t) + X_{t+1} + \cdots + X_{T-1} = 0,$$

i.e.,

$$D_t = 1 + \frac{X_{t+1}}{X_t} + \cdots + \frac{X_{T-1}}{X_t}.$$

In practical implementations, however, a zero-variance solution is not achievable due to non-availability of a continuum of duration required in the above formula as well as higher-order effects of the terminal wealth that we have not considered.

Appendix C: Analytical Heuristics for the Income Problem

In this section, we derive the analytical heuristics that allows us to think about the Income investor's objective and link it to the solution of the analytical mean variance problem. We observe that when the returns of the duration and the non-duration assets are jointly normally distributed, then the first-order terminal wealth which is a sum of normal random variables is also normally distributed. Let the weights in the non-duration assets through time be $\boldsymbol{w} = (w_0, w_1, \ldots, w_{T-1})$ and the durations of the duration assets through time be $\boldsymbol{D} = (D_0, D_1, \ldots, D_{T-1})$. Then the first-order terminal wealth subject to an additional income withdrawal A can be written as:

$$P_T^{(1)}(\boldsymbol{w}, \boldsymbol{D}, A) = \mu(\boldsymbol{w}, \boldsymbol{D}, A) + \sigma(\boldsymbol{w}, \boldsymbol{D}, A)z,$$

where z is a standard normal variable. Note that the Income investor's objective defined previously was to maximize the probability that the portfolio wealth does not fall below zero until the end of the investment horizon. However, to derive the analytical heuristics we use a simpler objective function which is to maximize the probability that terminal wealth is greater than zero. Mathematically, this simpler objective function is defined as:

$$J(\boldsymbol{W}) = Prob(P_T^{(1)}(\boldsymbol{w}, \boldsymbol{D}, A) \geq 0).$$

The objective function can be further simplified to:

$$P(P_T^{(1)}(\boldsymbol{w}, \boldsymbol{D}, A) \geq 0) = P\left(z \geq -\frac{\mu(\boldsymbol{w}, \boldsymbol{D}, A)}{\sigma(\boldsymbol{w}, \boldsymbol{D}, A)}\right) = \Phi\left(\frac{\mu(\boldsymbol{w}, \boldsymbol{D}, A)}{\sigma(\boldsymbol{w}, \boldsymbol{D}, A)}\right),$$

where Φ is the standard normal CDF. Since Φ is monotonically increasing, the above function is maximized when the ratio $\mu(\boldsymbol{w}, \boldsymbol{D}, A)/\sigma(\boldsymbol{w}, \boldsymbol{D}, A)$

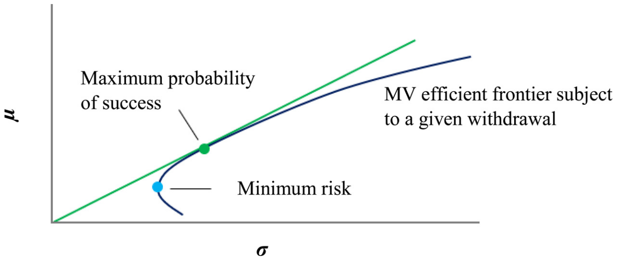

Figure C1 Maximum probability of success point lies on the mean–variance (MV) efficient frontier (EF).

is maximized. If we fix a certain $\sigma(\boldsymbol{w}, \boldsymbol{D}, A)$, the ratio is maximized when $\mu(\boldsymbol{w}, \boldsymbol{D}, A)$ is the corresponding y-value of the mean-variance efficient frontier. In terms of maximizing this ratio, any point below the efficient frontier can be discarded, i.e., the solution must lie on the efficient frontier. Furthermore, for all (x, y) pairs on the efficient frontier, the ratio y/x is maximized when the line from the origin to the (x, y) point forms a tangent line on the efficient frontier. This is demonstrated in Figure C1.

We highlight a key concept here which is the duality between the mean–variance optimization and the income optimization. We underscore the fact that for a given withdrawal level, the maximum probability of success solution is the point of tangency from the origin to the mean–variance efficient frontier. The analytical solution of the income optimization problem is nothing but the analytical solution of the corresponding tangent point on the mean–variance frontier.

Furthermore, for different levels of withdrawal $A_1 < A_2 < A_3 < \cdots < A_n < \cdots$, we can obtain a sequence of mean–variance efficient frontiers and subsequently pick a point on each of them that corresponds to the maximum probability of success for these income levels respectively. The sequence of the tangency points when plotted as one minus the probability of success (x-axis) versus withdrawal levels (y-axis) produces the income efficient frontier.

Remark (*Allocation stops changing materially around 50% success probability*): *As income withdrawal increases, expected values from the mean–variance efficient frontier decrease and at a certain income level the whole efficient frontier drops below zero. At this income level, the maximum probability of success is achieved on the furthest point of the mean–variance efficient frontier where the probability is* $\Phi(0)$, *i.e.*, *50%. On further increasing the income withdrawal, the efficient frontier keeps falling below the zero line*; *however*, *the maximum probability of success point is still the furthest point on the frontier. The allocation of the furthest point on the efficient frontiers below the zero line does not differ materially and is characterized by heavy allocation towards equities.*

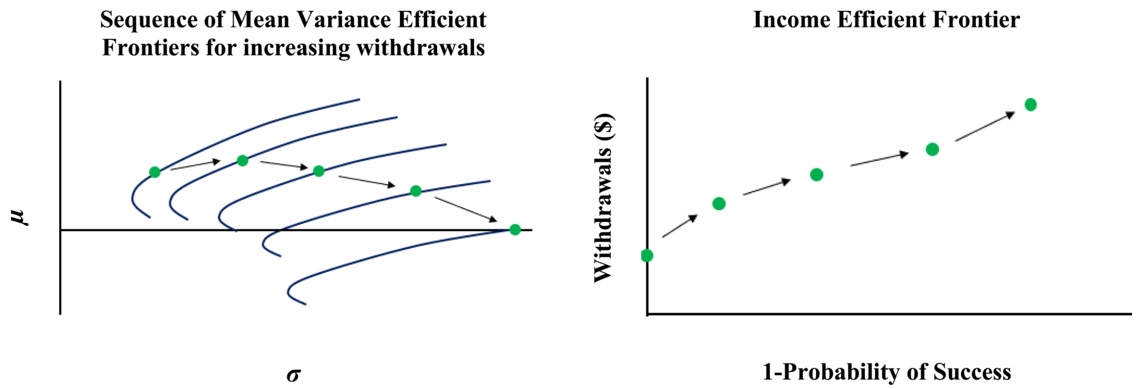

Figure C2 Income efficient frontier (right) as a solution of a sequence of mean variance solutions (left).

Appendix D: Analytical Versus Simulation-Based Results

For this exercise we simulate 200,000 paths based on our stylized two-process model from Section 5 using the same problem parameters (Growth Investor Scenario 2). We generate the return series of a growth asset and 30 constant maturity Treasury zero-coupon bond indexes with maturities of 1-year, 2-year, . . . , 30-year which are priced off of the simulated yield curve paths. The use of constant maturity indexes facilitates the process

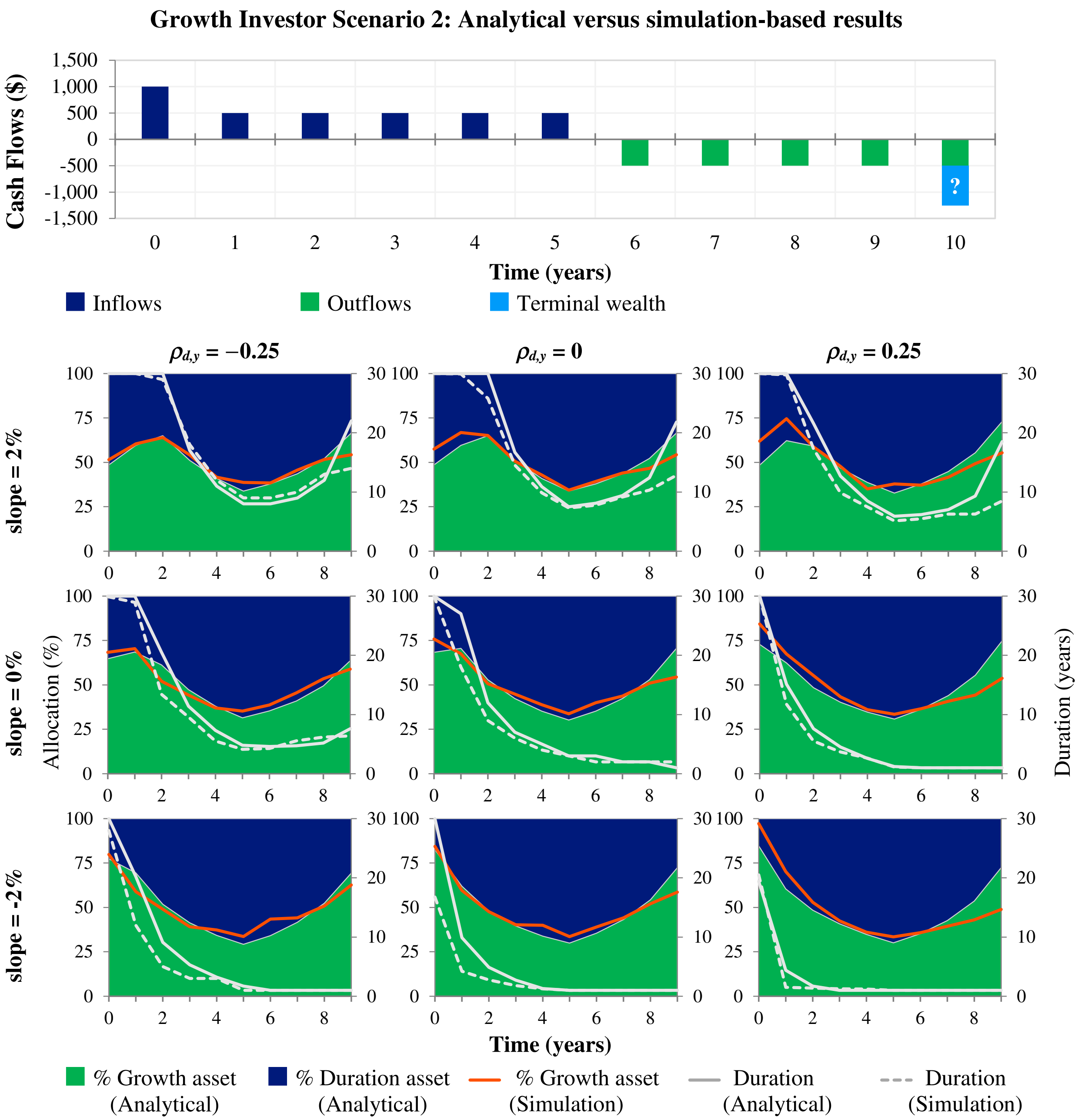

Figure D1 Cash flow schedule with terminal wealth at $t = 10$ to be determined along with a comparison of allocations and durations for the analytical and SBPS approaches under the simplistic **growth-duration** model.

of allocating to hold-to-maturity investments. While allocations are made to indexes, implementation is accomplished through investments in hold-to-maturity assets. Calculated durations are a function of the combination of the constant maturity indexes indicated by the solution which should closely match that of the corresponding hold-to-maturity assets. Figure D1 presents the simulation-based equity allocations and fixed-income durations through time for various correlation coefficients and yield curve slope scenarios along with the equity allocations and durations from our analytical framework.

Based on this analysis, we make the following observations:

- The simulation-based approach produces optimal allocation and duration outcomes that are closely aligned with our analytical results
- In general, the *Median vs Median-percentile* allocations start with higher equity allocations and end with lower equity than the analytical *Mean–Variance* allocation. This is so because in the analytical framework, the growth of the portfolio balance does not compound beyond the first order which results in lower balances and higher allocations to the risky asset.

Appendix E: Conversions Between Real and Nominal Dollars

A real cash flow is essentially a nominal cash flow discounted by the inflation index. Assume we have an inflation index I at time $t = 0$, $I = I_0$. Let $C_{t,N}$ and $C_{t,R}$ be a nominal and a real cash flow in future time t.

The nominal cash flow $C_{t,N}$ can be expressed in nominal and real terms as

$$C_{t,N}^{(N)} = C_{t,N} \quad \text{and} \quad C_{t,N}^{(R)} = C_{t,N}\frac{I_0}{I_t}.$$

Similarly, the real cash flow $C_{t,R}$ can be expressed in nominal and real terms by

$$C_{t,R}^{(N)} = C_{t,R}\frac{I_t}{I_0} \quad \text{and} \quad C_{t,R}^{(R)} = C_{t,R}.$$

The above identities allow us to unify nominal and real cash outflows in a single unit, i.e., either a nominal unit or a real unit. We can also use the above identities to go back and forth between nominal and real portfolio wealth and asset returns. Let the portfolio wealth at time t be expressed in nominal and real terms be $P_t^{(N)}$ and $P_t^{(R)}$, and let the nominal and real cash flows through time be $C_{t,N}$ and $C_{t,R}$ for $t = 0, 1, \ldots, T$. Then, portfolio wealth evolution in nominal and real terms can be described by:

$$P_{t+1}^{(N)} = P_t^{(N)}(1 + w_t^T r_t) + C_{t+1,N} + C_{t+1,R}\frac{I_{t+1}}{I_0}$$

and

$$P_{t+1}^{(R)} = P_t^{(R)}(1 + w_t^T r_t)\frac{I_t}{I_{t+1}} + C_{t+1,N}\frac{I_0}{I_{t+1}} + C_{t+1,R}.$$

Appendix F: Exploratory Statistics of Simulation Data

We summarize the mean arithmetic return (μ) and the standard deviation (σ) of returns of some key indexes at $t = 0, 1, 2, 3, 4, 5, 7, 10, 20, 30, 40$ in Table F1. In Table F2 we present the average correlation matrix of the assets. The average is taken over all correlation matrices observed at $t = 0, 1, 2, \ldots, 40$.

Disclaimer

The views expressed herein are those of the authors and do not necessarily represent the views of Invesco.

Table F1 Arithmetic means of 1-year simulated returns at different points in time through the investment horizon and the corresponding standard deviations.

Assets	Statistic	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$	$t = 7$	$t = 10$	$t = 20$	$t = 30$	$t = 40$
US equities	μ	6.8	6.6	6.5	6.7	6.8	7.1	7.3	7.6	8.1	8.4	8.5
	σ	20.2	20.7	20.7	20.8	20.8	20.9	20.9	21.1	21.2	21.3	21.3
Developed	μ	6.0	5.8	5.9	6.0	6.1	6.4	6.6	6.8	7.4	7.6	7.8
equities	σ	18.4	18.5	18.4	18.6	18.6	18.5	18.7	18.8	18.9	18.9	18.9
EM equities	μ	7.7	7.5	7.6	7.7	7.8	8.1	8.4	8.6	9.0	9.4	9.6
	σ	26.7	27.5	27.7	27.7	27.9	27.8	28.0	28.0	28.2	28.3	28.4
STRIPS 1-year	μ	1.9	1.7	1.8	1.9	2.1	2.2	2.5	2.8	3.3	3.5	3.7
	σ	0.0	1.1	1.4	1.6	1.8	2.0	2.2	2.5	3.0	3.2	3.3
STRIPS 5-year	μ	1.8	1.5	1.6	1.8	2.0	2.2	2.5	2.9	3.9	4.2	4.4
	σ	3.3	3.5	3.7	3.9	4.1	4.3	4.5	4.9	5.6	5.9	6.1
STRIPS 10-year	μ	1.9	1.5	1.6	1.7	1.9	2.1	2.4	2.9	3.9	4.5	4.8
	σ	5.4	5.7	5.9	6.1	6.3	6.5	6.8	7.2	8.2	8.6	8.8
STRIPS 20-year	μ	2.2	1.7	1.6	1.6	1.7	1.9	2.2	2.7	3.6	4.6	5.1
	σ	8.2	8.4	8.6	8.9	9.1	9.3	9.6	10.1	11.1	11.8	12.0
STRIPS 30-year	μ	2.4	1.8	1.6	1.5	1.5	1.7	2.1	2.6	3.3	4.5	5.1
	σ	9.9	10.0	10.3	10.5	10.7	10.9	11.3	11.7	12.7	13.6	13.9
TIPS 1-year	μ	1.9	1.7	1.8	1.9	2.1	2.2	2.5	2.8	3.3	3.5	3.7
	σ	0.9	1.4	1.7	1.9	2.0	2.2	2.4	2.6	3.2	3.4	3.5
TIPS 5-year	μ	1.9	1.6	1.6	1.8	1.9	2.1	2.4	2.9	3.8	4.2	4.4
	σ	3.7	3.9	4.0	4.1	4.1	4.3	4.3	4.5	4.9	5.0	5.1
TIPS 10-year	μ	2.0	1.6	1.6	1.8	1.9	2.1	2.5	3.0	4.1	4.6	4.8
	σ	5.7	5.8	5.9	6.0	6.0	6.1	6.2	6.3	6.6	6.7	6.7
TIPS 20-year	μ	2.1	1.7	1.7	1.8	2.0	2.2	2.6	3.1	4.4	5.1	5.3
	σ	10.0	10.0	10.1	10.2	10.2	10.3	10.3	10.4	10.7	10.9	10.9
TIPS 30-year	μ	2.2	1.7	1.7	1.9	2.0	2.2	2.6	3.1	4.4	5.1	5.4
	σ	11.0	10.9	11.0	11.0	11.0	11.1	11.1	11.2	11.4	11.5	11.5
Inflation	μ	1.4	1.3	1.5	1.7	1.8	1.8	1.9	2.0	2.2	2.3	2.5
	σ	0.9	1.4	1.7	1.9	2.0	2.2	2.4	2.6	3.2	3.3	3.4

Table F2 Average asset correlation matrix.

Assets	US equities	Developed equities	EM equities	STRIPS 1-year	STRIPS 5-year	STRIPS 10-year	STRIPS 20-year	STRIPS 30-year	TIPS 1-year	TIPS 5-year	TIPS 10-year	TIPS 20-year	TIPS 30-year	Inflation index
US equities	1	0.71	0.65	0.14	0.20	0.20	0.19	0.18	0.13	0.16	0.15	0.12	0.11	0.07
Developed equities	0.71	1	0.64	0.15	0.21	0.21	0.20	0.19	0.14	0.17	0.16	0.13	0.12	0.08
EM equities	0.65	0.64	1	0.11	0.21	0.22	0.22	0.22	0.10	0.16	0.15	0.13	0.13	0.06
STRIPS 1-year	0.14	0.15	0.11	1	0.54	0.38	0.27	0.23	0.94	0.59	0.43	0.27	0.25	0.52
STRIPS 5-year	0.20	0.21	0.21	0.54	1	0.95	0.82	0.75	0.50	0.67	0.56	0.40	0.37	0.28
STRIPS 10-year	0.20	0.21	0.22	0.38	0.95	1	0.95	0.90	0.35	0.64	0.58	0.45	0.42	0.19
STRIPS 20-year	0.19	0.20	0.22	0.27	0.82	0.95	1	0.99	0.25	0.59	0.58	0.49	0.47	0.14
STRIPS 30-year	0.18	0.19	0.22	0.23	0.75	0.90	0.99	1	0.22	0.56	0.56	0.49	0.47	0.12
TIPS 1-year	0.13	0.14	0.10	0.94	0.50	0.35	0.25	0.22	1	0.61	0.45	0.28	0.26	0.61
TIPS 5-year	0.16	0.17	0.16	0.59	0.67	0.64	0.59	0.56	0.61	1	0.94	0.78	0.74	0.37
TIPS 10-year	0.15	0.16	0.15	0.43	0.56	0.58	0.58	0.56	0.45	0.94	1	0.94	0.92	0.27
TIPS 20-year	0.12	0.13	0.13	0.27	0.40	0.45	0.49	0.49	0.28	0.78	0.94	1	1.00	0.17
TIPS 30-year	0.11	0.12	0.13	0.25	0.37	0.42	0.47	0.47	0.26	0.74	0.92	1.00	1	0.16
Inflation index	0.07	0.08	0.06	0.52	0.28	0.19	0.14	0.12	0.61	0.37	0.27	0.17	0.16	1

These values are obtained as the arithmetic average of the correlation matrices observed at $t = 0, 1, 2, \ldots, 40$.

Notes

[1] Duration-targeted investments refer to strategies that exhibit relatively stable duration characteristics. Duration targeting can be explicit or implicit. Most benchmark-aware bond funds implicitly target duration as a function of tracking a specific benchmark, such as an intermediate-term bond index, that is regularly rebalanced to meet specific maturity specifications. Consequently, standard fixed-income investments behave more like constant maturity or constant duration assets.

[2] The necessary conditions indicated by Markowitz were the following:

1 The investor owns only liquid assets;
2 he maximizes the expected value of the utility of consumption;
3 the set of available probability distributions remain the same through time.

[3] Only recently has computational power evolved to allow for the computation of dynamic programming solutions to more practical multi-period portfolio selection problems. However, the number of state variables remains an important consideration.

[4] The simplifying assumptions from Mossin/Samuelson/Merton were the following:

1 Asset returns are i.i.d.;
2 investors have constant relative risk aversion utility;
3 all assets are liquid and tradeable;
4 markets are frictionless and complete; and
5 utility depends only on terminal wealth

[5] Time-dependence is used to describe instances where the tax rate that applies to an investment is dependent on the length of time an investment is held. For example, in the US, capital gains for investments held for less than 1 year are taxed at a higher rate than if an investment is held for longer than 1 year.

[6] Over the last decade there have been significant advancements in hardware technologies with Nvidia and AMD introducing general purpose Graphics Processing Units (GPUs) around the mid-2000s. Nvidia then launched CUDA in 2006, a software development kit (SDK) and application programming interface (API) that allows C programs to be executed on GPUs. Since then, GPUs have come a long way in terms of speed and handling big data. In 2017, Google released TensorFlow 1.0, which implements large-scale machine learning algorithms that can leverage GPU hardware. This combination of advanced GPUs and machine learning libraries has allowed us to solve the multi-period portfolio selection problem using a simulation-based approach.

[7] For a normal distribution, the distance between the median and the 15th percentile is roughly equal to one standard deviation.

[8] To facilitate the exposition of the practical application of SBPS, we do not explicitly account for the taxation of assets in the solutions presented. However, the approach allows for the flexibility to allow for the incorporation of a model for asset taxation that can then be used to provide optimal after-tax solutions to multi-period portfolio selection problems.

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Keywords: Goals-based investing; multi-period portfolio selection; asset-liability management; wealth management; retirement planning; simulation; Adaptive Moment (Adam) Optimization.